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Classification of Lagrangian surfaces of constant curvature in complex projective plane

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Abstract

From Riemannian geometric point of view, one of the most fundamental problems in the study of Lagrangian submanifolds is the classification of Lagrangian immersions of real space forms into complex space forms. The purpose of this article is thus to classify Lagrangian surfaces of constant curvature in complex projective plane CP^2. Our main result states that there are 29 families of Lagrangian surfaces of constant curvature in CP^2. Twenty-two of the 29 families are constructed via Legendre curves. Conversely, Lagrangian surfaces of constant curvature in CP^2 are obtained from the 29 families. As an immediate by-product, many interesting new examples of Lagrangian surfaces of constant curvature in CP^2 are discovered.

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1. Introduction

A submanifold M of a Kaehler manifold $\tilde{M}$ is called Lagrangian if the almost complex structure J of $\tilde{M}$ interchanges each tangent space of M with its corresponding normal space.

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Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics. For instance, the systems of partial differential equations of Hamilton–Jacobi type lead to the study of Lagrangian submanifolds and foliations in the cotangent bundle. Furthermore, Lagrangian submanifolds are part of a growing list of mathematically rich special geometries that occur naturally in string theory.

For a Lagrangian submanifold M with mean curvature vector H and shape operator A, the dual 1-form of JH is the well-known Maslov form. A Lagrangian submanifold is called Maslovian if it has no minimal points and if its Maslov vector field JH is an eigenvector of A_H.

In the study of Lagrangian submanifolds, it is important to construct non-trivial new examples. Also, from Riemannian geometric point of view, one of the most fundamental problems in the study of Lagrangian submanifolds is to classify Lagrangian immersions of real space forms into complex space forms. Such submanifolds are either totally geodesic or flat if the immersions were minimal [8,11] (for indefinite case, this was done in a series of articles [9,12,13,15]). For non-minimal Lagrangian immersions, this problem have been studied in [3,4,6,7] among others. In particular, Lagrangian submanifolds of constant curvature c in complex space forms of holomorphic sectional curvature $4c$ have been determined in [6] by utilizing the notion of twisted products. Moreover, Maslovian Lagrangian immersions of real space forms into complex space forms were classified in [3,4,7]. In particular, Maslovian Lagrangian surfaces of constant curvature in complex projective space were classified in [4].

In this paper, we classify Lagrangian surfaces of constant curvature in the complex projective plane CP^2 without the Maslovian condition. The class of Lagrangian surfaces of constant curvature in CP^2 is much bigger than the class of Maslovian Lagrangian surfaces of constant curvature. In fact, our main result states that there are 29 families of Lagrangian surfaces of constant curvature in CP^2. Twenty-two of the 29 families are constructed via Legendre curves. Conversely, Lagrangian surfaces of constant curvature in CP^2 are obtained locally from the 29 families. As an immediate by-product, many interesting new examples of Lagrangian surfaces of constant curvature in CP^2 are discovered.

2. Preprimaries

Let $\tilde{M}^n(4c)$ denote a complete simply-connected Kaehler n-manifold $\tilde{M}^n(4c)$ with constant holomorphic sectional curvature $4c$ and M a Lagrangian submanifold in $\tilde{M}^n(4c)$. We denote the Riemannian connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given, respectively, by (cf. [1])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for tangent vector fields X, Y and normal vector field ξ, where D is the connection on the normal bundle. The second fundamental form h is related to the shape operator A_ξ by $\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$. The mean curvature vector H of M is defined by $H = (1/n)\,\mathrm{trace}\, h$. A point $p \in M$ is called minimal if H vanishes at p.

For Lagrangian submanifolds M in $\tilde{M}^n(4c)$ we have (cf. [8])

$$D_X JY = J\nabla_X Y, \tag{2.3}$$

$$\langle h(X,Y), JZ\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle. \tag{2.4}$$

If we denote the Riemann curvature tensor of M by R, then the equations of Gauss and Codazzi are given, respectively, by

$$\langle R(X,Y)Z, W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.5}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.6}$$

where X, Y, Z, W are tangent to M and ∇h is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.7}$$

We recall the following theorems for later use (cf. [5]).

Theorem A. *Let $(M^n, \langle\ .\ ,\ .\rangle)$ be an n-dimensional simply connected Riemannian manifold. Let σ be a TM-valued symmetric bilinear form on M satisfying*

(i) $\langle \sigma(X,Y), Z\rangle$ *is totally symmetric,*
(ii) $(\nabla\sigma)(X,Y,Z) = \nabla_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ *is totally symmetric,*
(iii) $R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y),$

Then there exists a Lagrangian isometric immersion $L : M \to \tilde{M}^n(4c)$ whose second fundamental form h is given by $h = J\sigma$.

Theorem B. *Let $L_1, L_2 : M \to \tilde{M}^n(4c)$ be Lagrangian isometric immersions of a Riemannian n-manifold with second fundamental forms h^1 and h^2, respectively. If*

$$\langle h^1(X,Y), JL_{1\star}Z\rangle = \langle h^2(X,Y), JL_{2\star}Z\rangle \tag{2.8}$$

for all vector fields X,Y,Z tangent to M, then there exists a biholomorphic isometry ϕ of $\tilde{M}^n(4c)$ such that $L_1 = L_2 \circ \phi$.

3. Lagrangian and Legendrian submanifolds

We recall the following basic relationship between Legendrian submanifolds of $S^{2n+1}(1)$ and Lagrangian submanifolds of the complex projective n-space $CP^n(4)$ with constant holomorphic curvature 4 (cf. [14]).

Let

$$S^{2n+1}(r) = \{(z_1, \ldots, z_{n+1}) \in \boldsymbol{C}^{n+1} : \langle z, z\rangle = r^2\}$$

be the hypersphere in $\boldsymbol{C}^{n+1}$ centered at the origin with radius r. On $\boldsymbol{C}^{n+1}$ we consider the complex structure J induced by $i = \sqrt{-1}$. On $S^{2n+1}(1)$ we consider the canonical Sasakian structure consisting of ϕ given by the projection of the complex structure J of $\boldsymbol{C}^{n+1}$ on the tangent bundle of $S^{2n+1}(1)$ and the structure vector field $\xi = Jx$ with x being the position vector.

An isometric immersion $f: M \to S^{2n+1}(1)$ is called *Legendrian*, if ξ is normal to $f_*(TM)$ and $\langle \phi(f_*(TM)), f_*(TM)\rangle = 0$, where $\langle , \rangle$ denotes the inner product on $\boldsymbol{C}^{n+1}$. The vectors of $S^{2n+1}(1)$ normal to ξ at a point z define the horizontal subspace $\mathcal{H}_z$ of the Hopf fibration $\pi: S^{2n+1}(1) \to CP^n(4)$. Therefore, the condition "ξ is normal to $f_*(TM)$" means that f is horizontal; thus it describes an integral manifold of maximal dimension of the contact distribution $\mathcal{H}$.

Let $\psi: M \to CP^n(4)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau: \hat{M} \to M$ and a Legendrian immersion $f: \hat{M} \to S^{2n+1}(1)$ such that $\psi(\tau) = \pi(f)$. Hence, every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a Legendrian immersion of the same Riemannian manifold.

Conversely, suppose that $f: \hat{M} \to S^{2n+1}(1)$ is a Legendrian immersion. Then $\psi = \pi(f): M \to CP^n(4)$ is again an isometric immersion, which is Lagrangian. Under this correspondence, the second fundamental forms h^f and h^ψ of f and ψ satisfy $\pi_* h^f = h^\psi$. Moreover, h^f is horizontal with respect to π. We shall denote h^f and h^ψ simply by h.

Let $L: M \to S^{2n+1}(1) \subset \boldsymbol{C}^{n+1}$ be an isometric immersion. Denote by $\hat{\nabla}$ and ∇ the Levi–Civita connections of $\boldsymbol{C}^{n+1}$ and M, respectively. Let h denote the second fundamental form of M in $S^{2n+1}(1)$. Then we have:

$$\hat{\nabla}_X Y = \nabla_X Y + h(X, Y) - \langle X, Y\rangle L. \tag{3.1}$$

4. Legendre curves

A curve $z = z(t)$ is called regular if its speed, $v(t) := |z'(t)|$, is nowhere zero. A regular curve $z = z(s)$ in the hypersphere $S^{2n-1}(r) \subset \boldsymbol{C}^{n+1}$ is called *Legendre*, if $\langle z'(t), iz(t)\rangle = 0$ holds identically.

The following lemma follows easily from the definition.

Lemma 1. *Every horizontal lift of a regular curve in a Lagrangian submanifold of $CP^n(4)$ via the Hopf fibration π is a Legendre curve in $S^{2n+1}(1) \subset \boldsymbol{C}^{n+1}$.*

It is known that a unit speed Legendre curve $z(s)$ in $S^3(r) \subset \boldsymbol{C}^2$ satisfies:

$$z''(s) = i\kappa(s)z'(s) - \frac{z(s)}{r^2}, \tag{4.1}$$

where $\kappa(s)$ is the curvature function of z in $S^3(1)$ (cf. [4]).

A unit speed curve $z(s)$ in $S^3(r) \subset \boldsymbol{C}^2$ satisfies $z''(s) = -z(s)/r^2$ if and only if it is a geodesic. A geodesic in $S^3(r)$ can be Legendre or non-Legendre. For examples, $z(s) = (\cos s, \sin s)$ is Legendre and $z(s) = (e^{is}, 0)$ is non-Legendre.

Lemma 2. *Let r be a positive number r. Then we have*

(1) *Every Legendre curve* $z = z(t) : I \to S^3(r) \subset \boldsymbol{C}^2$ *satisfies*

$$z''(t) = i\lambda(t)z'(t) + \frac{v'}{v}z'(t) - \frac{v^2}{r^2}z(t), \tag{4.2}$$

where v is the speed of z and $\lambda = \kappa v$ *with* κ *being the curvature of z in* $S^3(r)$.

(2) *Conversely, if a regular curve* $z = z(t)$ *in* $S^3(r) \subset \boldsymbol{C}^2$ *with speed v satisfying differential* Eq. (4.2) *for some nowhere zero real-valued function* λ, *then z is a Legendre curve.*

Let $S^2(1/2) := \{(x_1, x_2, x_3) \in \boldsymbol{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1/4\}$. Then the Hopf fibration $\pi : S^3(1) \to CP^1(4) \equiv S^2(1/2)$ is given by

$$\pi(z, w) = \left(z\bar{w}; \frac{|z|^2 - |w|^2}{2}\right), \qquad (z, w) \in S^3(1) \subset \boldsymbol{C}^2. \tag{4.3}$$

For each Legendre curve γ in $S^3(1) \subset \boldsymbol{C}^2$, the projection $\pi \circ \gamma$ is a regular curve in $S^2(1/2)$. Conversely, each regular curve ξ in $S^2(1/2)$ gives rise to a horizontal lift in $S^3(1)$ which is unique up to a factor $e^{i\theta}$, $\theta \in \boldsymbol{R}$. Each horizontal lift of ξ is a Legendre curve in $S^3(1)$. Since the Hopf fibration π is a Riemannian submersion, a Legendre curve γ in S^3 is projected to a curve in $S^2(1/2)$ with the same curvature.

In order to explain our methods for constructing Lagrangian surfaces of constant curvature in CP^2, we recall the notions of special Legendre curves and their associated special Legendre curves which are introduced in [2,4].

Let $z = z(t)$ be a Legendre curve in $S^5(r) \subset \boldsymbol{C}^3$ with speed v. Then z, iz, z', iz' are orthogonal vector fields. Differentiating $\langle z'(t), iz(t)\rangle = \langle z'(t), z(t)\rangle = 0$ yields $\langle z'', iz\rangle = 0$ and $\langle z'', z\rangle = -1$. Thus, there exists a non-zero normal vector field P_z perpendicular to z, iz, z', iz' such that

$$z''(t) = i\psi(t)z'(t) + \frac{v'}{v}z'(t) - \frac{v^2}{r^2}z(t) - a(t)P_z(t) \tag{4.4}$$

for some real-valued functions ψ and a. The Legendre curve z is called *special* if P_z in (4.4) is a parallel normal vector field, i.e., $P_z'(t) = a(t)z'(t)$ for some function $a(t)$. If a special Legendre curve does not lie in any proper linear complex subspace of $\boldsymbol{C}^3$, then the function a in (4.4) is not identical zero. If z is a special Legendre curve satisfying (4.4), then P_z is also a special Legendre curve on $I' = \{t \in I : a(t) \neq 0\}$ which is called the *associated special Legendre curves* of z (see [4]).

Special Legendre curves do exist extensively. In fact, it was proved in [2] that, for any given non-zero functions $\psi(s), a(s)$ defined on an open interval I, there exists a unit speed special Legendre curve $z : I \to S^5(c) \subset \boldsymbol{C}^3$ satisfying (4.4) with $|P_z| = 1$.

5. Classification theorem

The main result of this paper is following classification theorem.

Theorem 1. *There are 29 families of Lagrangian surfaces of constant curvature in the complex projective plane $CP^2(4)$:*

(1) *Totally geodesic Lagrangian surfaces of constant curvature one.*

(2) *Flat minimal Lagrangian surface defined by*

$$L(s,t)=\frac{1}{\sqrt{3}}\left(e^{-i\sqrt{2}s},\ \sqrt{2}e^{is/\sqrt{2}}\cos(\sqrt{3/2}\,t),\ \sqrt{2}e^{is/\sqrt{2}}\cos(\sqrt{3/2}\,t)\right).$$

(3) *Lagrangian surfaces of curvature one defined by $\pi\circ L$ with*

$$L(s,y)=(\cos y,\ z(s)\sin y),$$

where $z(s)=(z_1(s),z_2(s))$ is an arbitrary unit speed Legendre curve in $S^3(1)\subset \boldsymbol{C}^2$.

(4) *Lagrangian surfaces of curvature one defined by $\pi\circ L$ with*

$$L(s,y)=z(s)\cos y+P_z(s)\sin y,$$

where $z: I\to S^5(1)\subset \boldsymbol{C}^3$ is an arbitrary unit speed special Legendre curve and $P_z: I\to S^5(1)\subset \boldsymbol{C}^3$ is the associated special Legendre curve of z.

(5) *Lagrangian surfaces of positive curvature c^2 defined by $\pi\circ L$ with*

$$L(s,y)=e^{i(b-c)s}z(y)+e^{i(b+c)s}w(y),\qquad b>0,\qquad c=\sqrt{1+b^2},$$

where $z: I\to S^5(\sqrt{b+c}/\sqrt{2c})\subset \boldsymbol{C}^3$ is an arbitrary special Legendre curve with speed $1/2$ and $w: I\to S^5(\sqrt{c-b}/\sqrt{2c})$ is the associated special Legendre curve of z with speed $1/2$.

(6) *Lagrangian surfaces of positive curvature a^2 defined by $\pi\circ L$, where*

$$L(s,y)=(z_1(s),\ z_2(s)\cos y,\ z_2(s)\sin y)$$

and $z=(z_1,z_2)$ is a unit speed Legendre curve in $S^3(1)\subset \boldsymbol{C}^2$ given by

$$z_1=\frac{e^{ibs}}{2a}[(a-b)e^{ias}-(a+b)e^{-ias}],$$

$$z_2=\frac{e^{ibs}\cos(as)}{a},\qquad a=\sqrt{1+b^2}$$

for an arbitrary non-zero real number b.

(7) *The Lagrangian surface of curvature one defined by $\pi\circ L$, where*

$$L(s,y)=(z_1(s),z_2(s)\cos y,z_2(s)\sin y)$$

and $z=(z_1,z_2)$ is the unit speed Legendre curve in $S^3(1)\subset \boldsymbol{C}^2$ given by

$$z_1=\frac{1-i\sin s}{\sqrt{2}},\qquad z_2=\frac{\cos s}{\sqrt{2}}(\sec s+\tan s)^i.$$

(8) *The flat Lagrangian surface defined by* $\pi \circ L$, *where*

$$L(s, y) = (z_1(s), z_2(s) \cos y, z_2(s) \sin y)$$

and $z = (z_1, z_2)$ *is the unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{\sqrt{b^2 - 1}}{b e^{is/\sqrt{b^2-1}}}, \qquad z_2 = \frac{e^{i\sqrt{b^2-1}s}}{b}, \qquad b > 1.$$

(9) *The flat Lagrangian surface defined by* $\pi \circ L$, *where*

$$L(s, y) = (z_1(s), z_2(s) \cos\, y, z_2(s) \sin\, y)$$

and $z = (z_1, z_2)$ *is the unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{\sqrt{2 - s^2}}{\sqrt{2}} e^{i\sqrt{1-s^2} - i \tan^{-1}(\sqrt{1-s^2})}, \qquad z_2 = \frac{s}{\sqrt{2}} \left(\frac{s e^{\sqrt{1-s^2}}}{1 + \sqrt{1 - s^2}} \right)^i.$$

(10) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$, *where*

$$L(s, y) = (z_1(s), z_2(s) \cos y, z_2(s) \sin y)$$

and $z = (z_1, z_2)$ *is a unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = (b - i\sqrt{e^{-2bs} - k^2}) e^{bs + ikb^{-1} \tan^{-1}(k^{-1}\sqrt{e^{-2bs} - k^2})},$$

$$z_2 = e^{bs + ikb^{-1} \tan^{-1}(k^{-1}\sqrt{e^{-2bs}-k^2}) - ib^{-1}\sqrt{e^{-2bs}-k^2}}$$

with arbitrary positive number b and $k = \sqrt{b^2 + 1}$.

(11) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$, *where*

$$L(s, y) = (z_1(s), z_2(s) \cos y, z_2(s) \sin y)$$

and $z = (z_1, z_2)$ *is a unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{\cos(bs) \left(\sqrt{a^2 \cos^2(bs) - 1} + ia \sin(bs) \right)^{a/b}}{a(a^2 - 1)^{a/2b} e^{ib^{-1} \tan^{-1}(\sin(bs)/\sqrt{a^2 \cos^2(bs) - 1})}},$$

$$z_2 = \frac{\sqrt{a^2 - \cos^2(bs)} \left(\sqrt{a^2 \cos^2(bs) - 1} + ia \sin(bs) \right)^{a/b}}{a(a^2 - 1)^{a/2b} e^{i \tan^{-1}(b \sin(bs)/\sqrt{a^2 \cos^2(bs) - 1})}},$$

with arbitrary positive number $b > \sqrt{2}$ *and* $a = \sqrt{b^2 - 1} > |\sec(bs)|$.

(12) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$*, where*

$$L(s, y) = (z_1(s), z_2(s)\cos y, z_2(s)\sin y)$$

and $z = (z_1, z_2)$ *is a unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{\sqrt{1 + b^2 - \cos^2(bs)}\left(\sqrt{a^2\cos^2(bs) + 1} + ia\sin(bs)\right)^{a/b}}{b^{a/b}\sqrt{b^2+1}e^{i\tan^{-1}(b\sin(bs)/\sqrt{a^2\cos^2(bs)+1})}},$$

$$z_2 = \frac{\cos(bs)\left(\sqrt{a^2\cos^2(bs) + 1} + ia\sin(bs)\right)^{a/b}}{b^{a/b}\sqrt{b^2+1}e^{-ib^{-1}\tanh^{-1}(\sin(bs)/\sqrt{a^2\cos^2(bs)+1})}},$$

with arbitrary positive number $b > 1$ *and* $a = \sqrt{b^2 - 1}$.

(13) *Lagrangian surfaces of positive curvature* $b^2 < 1$ *defined by* $\pi \circ L$*, where*

$$L(s, y) = (z_1(s), z_2(s)\cos y, z_2(s)\sin y)$$

and $z = (z_1, z_2)$ *is a unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{\sqrt{b^2 + \sin^2(bs)}e^{-i\tan^{-1}\left(b\sin(bs)/\sqrt{1-a^2\cos^2(bs)}\right)}}{\sqrt{b^2+1}\left(i(\sqrt{1 - a^2\cos^2(bs)} + a\sin(bs))\right)^{ia/b}},$$

$$z_2 = \frac{e^{ib^{-1}\tanh^{-1}(\sin(bs)/\sqrt{1-a^2\cos^2(bs)})}\cos(bs)}{\sqrt{b^2+1}\,e^{iab^{-1}\sinh^{-1}(ab^{-1}\sin(bs))}},$$

with arbitrary positive number $b \in (0, 1)$ *and* $a = \sqrt{1 - b^2}$.

(14) *Lagrangian surface of negative curvature* $-b^2$ *defined by* $\pi \circ L$*, where*

$$L(s, y) = (z_1(s), z_2(s)\cos y, z_2(s)\sin y)$$

and $z = (z_1, z_2)$ *is a unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{i\sqrt{a^2 - k^2\cosh^2(bs)} - b\sinh(b)}{\sqrt{a^2 - b^2}e^{ib^{-1}k\sin^{-1}\left(k\sinh(bs)/\sqrt{a^2-k^2}\right)}},$$

$$z_2 = \frac{e^{iab^{-1}\tan^{-1}\left(a\sinh(bs)/\sqrt{a^2-k^2\cosh^2(bs)}\right)}\cosh(bs)}{\sqrt{a^2 - b^2}e^{ib^{-1}k\sin^{-1}\left(k\sinh(bs)/\sqrt{a^2-k^2}\right)}}$$

with arbitrary positive number b and $a > k := \sqrt{1 + b^2}$.

(15) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$, *where*

$$L(s, y) = (z_1(s), z_2(s) \cos y, z_2(s) \sin y)$$

and $z = (z_1, z_2)$ *is a unit speed Legendre curve in* $S^3(1) \subset \boldsymbol{C}^2$ *given by*

$$z_1 = \frac{\sinh(bs)\left(k \cosh(bs) + i\sqrt{a^2 - k^2 \sinh^2(bs)}\right)^{k/b}}{\sqrt{a^2+b^2}(1+a^2+b^2)^{k/2b} e^{iab^{-1} \tanh^{-1}(a \cosh(bs)/\sqrt{a^2-k^2 \sinh^2(bs)})}},$$

$$z_2 = \frac{(i\sqrt{a^2-k^2 \sinh^2(bs)} - b \cosh(bs))\left(k \cosh(bs) + i\sqrt{a^2-k^2 \sinh^2(bs)}\right)^{k/b}}{\sqrt{a^2+b^2}(1+a^2+b^2)^{k/2b}}$$

with arbitrary positive numbers a, b and $k = \sqrt{1+b^2}$.

(16) *Flat Lagrangian surfaces defined by* $\pi \circ L$, *with*

$$L(s, v) = \left(\frac{ae^{-is/a}}{\sqrt{1+a^2}}, z(v)e^{ias}\right), \qquad 0 \neq a \in \boldsymbol{R},$$

where $z = z(v)$ *is an arbitrary unit speed Legendre curve in* $S^3(1/\sqrt{1+a^2}) \subset \boldsymbol{C}^2$.

(17) *Flat Lagrangian surface defined by* $\pi \circ L$ *with*

$$L(s, v) = e^{i\sqrt{b^2-s^2}}\left(\frac{z(v)s^{1+ib}}{(c+\sqrt{b^2-s^2})^{ic}}, \frac{\sqrt{1+b^2-s^2}}{\sqrt{1+b^2}e^{i \tan^{-1}\sqrt{b^2-s^2}}}\right)$$

where b is an arbitrary positive number and $z = z(v)$ *is an arbitrary unit speed Legendre curve in* $S^3(1/\sqrt{1+b^2}) \subset \boldsymbol{C}^2$.

(18) *Lagrangian surfaces of positive curvature* c^2 *defined by* $\pi \circ L$, *with*

$$L = e^{ibs}\left(i \sin(cs) - \frac{b}{c} \cos(cs), 2z(t) \cos(cs)\right), \; c = \sqrt{1+b^2},$$

where b is an arbitrary positive number and $z = z(v)$ *is an arbitrary Legendre curve of speed* $1/2$ *in* $S^3(\frac{1}{2c}) \subset \boldsymbol{C}^2$.

(19) *Lagrangian surfaces of curvature one defined by* $\pi \circ L$, *with*

$$L(s, t) = \left(\frac{\sqrt{1+a^2 \sin^2 s}}{\sqrt{1+a^2}} e^{-i \tan^{-1}(a \sin s)}, z(t)(\sec s + \tan s)^{i/a} \cos s\right),$$

where a is an arbitrary positive number and $z = z(t)$ is an arbitrary Legendre curve of constant speed a in $S^3(a/\sqrt{1+a^2}) \subset \boldsymbol{C}^2$.

(20) *Lagrangian surfaces of positive curvature b^2 defined by $\pi \circ L$ with*

$$L(s,t) = \left(\frac{\sqrt{b^2 - c^2 - \cos^2 bs}(\sqrt{a^2\cos^2 bs - c^2} + ia\sin bs)^{a/b}}{\sqrt{b^2-c^2}\,\exp\left(i\tan^{-1}(b\sin(bs/\sqrt{a^2\cos^2 bs - c^2}))\right)}, \right.$$
$$z(t)\cos(bs)\exp i\left\{\frac{a}{b}\sin^{-1}\left(a\sin(bs)/\sqrt{a^2-c^2}\right)\right.$$
$$\left.\left. - \frac{c}{b}\tan^{-1}(c\tan(bs)/\sqrt{a^2 - c^2\sec^2 bs})\right\}\right),$$

where b, c are any positive numbers with $b > 1$, $z = z(t)$ is an arbitrary unit speed Legendre curve in $S^3(1/\sqrt{b^2-c^2}) \subset \boldsymbol{C}^2$ and $a = \sqrt{b^2-1}$.

(21) *Lagrangian surfaces of positive curvature b^2 defined by $\pi \circ L$ with*

$$L(s,t) = \left(\frac{\sqrt{b^2 + c^2 - \cos^2 bs}(\sqrt{a^2\cos^2 bs + c^2} + ia\sin bs)^{a/b}}{\sqrt{b^2+c^2}(a^2+c^2)^{a/2b}\exp(i\tan^{-1}((b\sin bs)/\sqrt{a^2\cos^2 bs + c^2}))}, \right.$$
$$z(t)(\cos bs)(\sqrt{a^2\cos^2 bs + c^2} + ia\sin bs)^{a/b}$$
$$\left. \times \exp\left\{\frac{ic}{b}\tanh^{-1}\left(\frac{c\tan bs}{\sqrt{a^2 + c^2\sec^2 bs}}\right)\right\}\right),$$

where b is an arbitrary positive number > 1, $z = z(t)$ an arbitrary Legendre curve of constant speed $(a^2+c^2)^{-a/2b}$ in $S^3(1/\sqrt{b^2+c^2}) \subset \boldsymbol{C}^2$, and $a = \sqrt{b^2-1}$.

(22) *Lagrangian surfaces of positive curvature b^2 defined by $\pi \circ L$ with*

$$L(s,t) = \left(\frac{\sqrt{c^2+\sin^2 s}}{\sqrt{1+c^2}} e^{-i\tan^{-1}((\sin s)/c)},\ z(t)e^{2ic\tanh^{-1}(\tan(s/2))}\cos s \right),$$

where $z = z(t)$ is an arbitrary unit speed Legendre curve in $S^3(\frac{1}{\sqrt{1+c^2}}) \subset \boldsymbol{C}^2$.

(23) *Lagrangian surfaces of positive curvature b^2 defined by $\pi \circ L$ with*

$$L(s,t) = \left(z(t)\exp i\left\{\frac{c}{b}\tanh^{-1}\left(\frac{c\sin bs}{\sqrt{c^2 - a^2\cos^2 bs}}\right)\right.\right.$$
$$\left. - \frac{a}{b}\sinh^{-1}\left(\frac{a\sin bs}{\sqrt{c^2-a^2}}\right)\right\}\cos bs,$$

$$\left.\frac{\sqrt{b^2+c^2-\cos^2 bs}\exp i\left\{\frac{a^2(2b^2+2c^2-1)}{2b^2(a^2-c^2)}\tan^{-1}\frac{\sqrt{c^2-a^2\cos^2 bs}}{b\sin bs}\right\}}{\sqrt{b^2+c^2}(a\sin bs+\sqrt{c^2-a^2\cos^2 bs})^{ia/b}\exp i\left\{\frac{a^2-2c^2}{2b^2(a^2-c^2)}\times\tan^{-1}\frac{b\sin bs}{\sqrt{c^2-a^2\cos^2 bs}}\right\}}\right),$$

where $z = z(t)$ is an arbitrary unit speed Legendre curve in $S^3(1/\sqrt{b^2+c^2}) \subset \boldsymbol{C}^2$, b is an arbitrary real number in $(0, 1)$ *and* $a = \sqrt{1-b^2}$.

(24) *Lagrangian surfaces of negative curvature $-b^2$ defined by $\pi \circ L$ with*

$$L(s,t) = e^{-ib^{-1}a\sin^{-1}\left(a(\sinh bs)/\sqrt{c^2-a^2}\right)} \times \left(\frac{i\sqrt{c^2-a^2\cosh^2 bs}-b\sinh bs}{\sqrt{c^2-b^2}},\right.$$
$$\left. z(t)e^{icb^{-1}\tan^{-1}\left(c(\sinh bs)/\sqrt{c^2-a^2\cosh^2 bs}\right)}\cosh bs\right),$$
$$a = \sqrt{1+b^2},$$

where $z = z(t)$ is an arbitrary unit speed Legendre curve in $S^3(1/\sqrt{c^2-b^2}) \subset \boldsymbol{C}^2$, b an arbitrary positive number, and c an arbitrary number $> a$.

(25) *Lagrangian surfaces of curvature one defined by $\pi \circ L$ with*

$$L(s,t) = \frac{\operatorname{sech} bs}{\sqrt{1+4b^2}}\left(\sqrt{(1+4b^2)\cosh^2 bs-4b^2}\,e^{-i\tan^{-1}(2b\tanh bs)},\right.$$
$$\left. 2be^{is/2}\cos\left(\sqrt{1+4b^2}\,t/2\right), 2be^{is/2}\sin\left(\sqrt{1+4b^2}\,t/2\right)\right),$$

where b is an arbitrary non-zero real number.

(26) *Curvature one Lagrangian surfaces of type $(A_{\rho\psi}, \alpha_{\rho\psi})$ described in Proposition* 6.1.

(27) *Lagrangian surfaces with positive curvature K of type $(B^K_{\mu\Phi}, \beta^K_{\mu\Phi})$ described in Proposition* 6.2.

(28) *Lagrangian surfaces of positive curvature K of type $(C^K_{\mu\Phi}, \gamma^K_{\mu\Phi})$ described in Proposition* 6.3.

(29) *Lagrangian surfaces of positive curvature K of type $(D^K_{\mu\Phi}, \delta^K_{\mu\Phi})$ described in Proposition* 6.4.

Conversely, up to rigid motions of $CP^2(4)$, locally (in a neighborhood of each point belonging to an open dense subset), every Lagrangian surface of constant curvature in $CP^2(4)$ is obtained from one of the 29 families.

Proof. Let M be a Lagrangian surface of constant curvature K in $CP^2(4)$. Denote the tangent bundle of M by TM. If M is minimal in $CP^2(4)$, then it is totally geodesic or flat (cf. [8,11]).

So, M is an open portion of a Lagrangian totally geodesic real projective plane $RP^2(1)$ or a flat minimal surface in $CP^2(4)$. This gives Cases (1) and (2). □

Next, let us assume that M is non-minimal. Then $U := \{p \in M : H(p) \neq 0\}$ is a non-empty open subset. We shall work on U instead of M.

For each point $p \in U$, we define a function γ_p by

$$\gamma_p : T^1_pU \to \mathbf{R} : v \mapsto \gamma_p(v) = \langle h(v, v), Jv\rangle,$$

where $T^1_pU := \{v \in T_pU : \langle v, v\rangle = 1\}$. Since T^1_pU is a unit circle which is compact, there exists a vector $v \in T^1_pU$ such that γ_p attains an absolute minimum at v. Since p is a non-totally geodesic point, (2.4) implies that $\gamma_p \neq 0$. So, by applying linearity, we have $\gamma_p(v) < 0$. As γ_p attains an absolute minimum at v, it follows from (2.4) that $\langle h(v, v), Jw\rangle = 0$ for all w orthogonal to v. Combining this with (2.4) shows that v is an eigenvector of the shape operator A_{Jv}. Hence, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM with $e_1 = v$ which satisfies

$$h(e_1, e_1) = \lambda Je_1, \qquad h(e_1, e_2) = \mu Je_2, \qquad h(e_2, e_2) = \mu Je_1 + \varphi Je_2, \tag{5.1}$$

for some functions λ, μ, φ. As $H \neq 0$, we have $(\lambda + \mu)^2 + \varphi^2 > 0$ on U.

If $\varphi = 0$ on U, the the Lagrangian surface is Maslovian. Hence, it follows from Theorem 3 of [4] that the Lagrangian surface, restricted to U, is given by one of the Lagrangian surfaces given in Cases (3)–(15).

Next, let us assume that $\varphi \neq 0$ on an open subset $V \subset U$. In this case, (5.1) and the equation of Codazzi imply

$$\begin{aligned} &e_1\mu = \varphi\omega_1^2(e_1) + (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_2\lambda = (\lambda - 2\mu)\omega_1^2(e_1), \\ &e_2\mu - e_1\varphi = 3\mu\omega_1^2(e_1) + \varphi\omega_1^2(e_2), \end{aligned} \tag{5.2}$$

where $\nabla_X e_1 = \omega_1^2(X)e_2$. Also, from (5.1) and the equation of Gauss we have

$$K = \lambda\mu - \mu^2 + 1 = \text{const.} \tag{5.3}$$

Case (I). $\nabla_{e_1}e_1 = 0$ *on an open neighborhood* V_1 *of a point in* V. In this case, (5.2) and (5.3) imply

$$e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_2\lambda = 0, \qquad e_2\mu - e_1\varphi = \varphi\omega_1^2(e_2) \tag{5.4}$$

on V_1. By differentiating (5.3) with respect to e_2 and by applying (5.4), we obtain $(\lambda - 2\mu)e_2\mu = 0$. Thus, we have either $\lambda = 2\mu$ or $e_2\mu = 0$ at each point of V_1.

If $\lambda = 2\mu$ on some connected open subset $W \subset V_1$, then $K = \mu^2 + 1$ on W which implies that μ is constant on W. So, $e_2\mu = 0$ also holds on W. Consequently, we have $e_2\mu = 0$ identically on V_1. Therefore, we obtain from (5.4) that

$$e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2), \qquad e_2\lambda = e_2\mu = 0, \qquad e_1\varphi = -\varphi\omega_1^2(e_2). \tag{5.5}$$

As we have $\nabla_{e_1}e_1 = 0$ on V_1, there exists a local coordinate system $\{s, u\}$ on V_1 such that the metric tensor is given by

$$g = \mathrm{d}s \otimes \mathrm{d}s + G^2(s, u)\,\mathrm{d}u \otimes \mathrm{d}u \tag{5.6}$$

for some function G with $\partial/\partial s = e_1$, $\partial/\partial u = Ge_2$. From (5.5) we have $\lambda = \lambda(s)$ and $\mu = \mu(s)$. Also, it follows from (5.6) that

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial s} = (\ln G)_s\frac{\partial}{\partial u}, \qquad \omega_1^2(e_2) = \frac{G_s}{G}. \tag{5.7}$$

By (5.5), (5.6) and (5.7), we find $(\ln G)_s = -(\ln\varphi)_s$. Thus, (5.6) becomes

$$g = \mathrm{d}s\otimes\mathrm{d}s + \frac{F^2(u)}{\varphi^2}\,\mathrm{d}u\otimes\mathrm{d}u, \qquad e_1 = \frac{\partial}{\partial s}, \qquad e_2 = \frac{\varphi}{F(u)}\frac{\partial}{\partial u} \tag{5.8}$$

for some positive function $F(u)$. By applying (5.8) and the equation of Gauss, we have $\varphi\varphi_{ss} - 2\varphi_s^2 = K\varphi^2$. After solving this differential equation, we obtain

$$\varphi = \begin{cases} A(u)\sec(bs + B(u)), & \text{if } K = b^2 > 0,\\ \dfrac{A(u)}{cs + B(u)}, & \text{if } K = 0,\\ A(u)\operatorname{sech}(bs + B(u)), & \text{if } K = -b^2 < 0, \end{cases} \tag{5.9}$$

for some functions $A(u)$, $B(u)$ and $b > 0$, where A is nowhere zero on V_1.

Let $t = t(u)$ be an antiderivative of $F(u)/A(u)$. Then (5.8) and (5.9) give

$$g = \begin{cases} \mathrm{d}s\otimes\mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t\otimes\mathrm{d}t, & \text{if } K = b^2 > 0,\\ \mathrm{d}s\otimes\mathrm{d}s + (cs + \theta(t))^2\,\mathrm{d}t\otimes\mathrm{d}t, & \text{if } K = 0,\\ \mathrm{d}s\otimes\mathrm{d}s + \cosh^2(bs + \theta(t))\,\mathrm{d}t\otimes\mathrm{d}t, & \text{if } K = -b^2 < 0, \end{cases} \tag{5.10}$$

some function $\theta(t)$.

We divide Case (I) into several cases.

Case (I.i). $\lambda = 2\mu$ *on an open subset* $U_1 \subset V_1$. In this case, both λ, μ are constant and $K = 1 + \mu^2 \geq 1$ on U_1 by (5.3).

If $\lambda = \mu = 0$ on U_1, the Lagrangian surface is Maslovian. So, this reduces to previous case. Hence, we may assume that $\lambda = 2\mu = 2b$ for some positive number b on U_1 which gives $K = 1 + b^2 > 1$. From (5.9) and (5.10) we have

$$\begin{aligned} &g = \mathrm{d}s\otimes\mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t\otimes\mathrm{d}t,\\ &\lambda = 2\mu = 2b > 0, \qquad \varphi = f(t)\sec(bs + \theta(t)),\\ &\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = -b\tan(bs + \theta(t))\frac{\partial}{\partial t},\\ &\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = \frac{b}{2}\sin(2bs + 2\theta(t))\frac{\partial}{\partial s} - \theta'(t)\tan(bs + \theta(t))\frac{\partial}{\partial t}, \end{aligned} \tag{5.11}$$

where f is non-zero function. Applying (5.1), (5.11) and the formula of Gauss, we find

$$\begin{aligned} &L_{ss} = 2ibL_s - L, \qquad L_{st} = (ib - c\tan(bs + \theta))L_t,\\ &L_{tt} = (ib\cos(bs + \theta(t)) + b\sin(bs + \theta(t)))\cos(bs + \theta(t))L_s\\ &\qquad + (if(t) - \theta'\tan(bs + \theta))L_t - \cos^2(bs + \theta(t))L. \end{aligned} \tag{5.12}$$

After solving the first equation of this system, we obtain

$$L = e^{i(b-c)s}(A(t) + B(t)e^{2ics}), \qquad c = \sqrt{1+b^2} \tag{5.13}$$

for some $\boldsymbol{C}^3$-valued functions $A(t)$, $B(t)$. By substituting this into the second equation of (5.12), we discover that $B'(t) = A'(t)e^{2i\theta}$. Hence, (5.13) becomes

$$L_t = A'(t)e^{i(b-c)s}(1 + e^{2i(bs+\theta)}). \tag{5.14}$$

If θ is constant on U_1, we have $\theta = 0$ after applying a suitable translation on s. Thus, (5.14) becomes $L_t = A'(t)e^{i(b-c)s}(1 + e^{2ics})$ which implies that

$$L = A(t)e^{i(b-c)s}(1 + e^{2ics}) + K(s) \tag{5.15}$$

for some $\boldsymbol{C}^3$-valued function $K(s)$. Substituting (5.15) into the first equation in (5.12) yields $K'' = 2ibK' - K$. Hence, after solving the last equation, we obtain $K(s) = e^{i(b-c)s}(a_1 + a_2e^{2ics})$ for some vectors $a_1, a_2 \in \boldsymbol{C}^3$. Therefore, we may put

$$L = F(t)(e^{i(b-c)s} + e^{i(b+c)s}) + c_1e^{i(b+c)s}$$

for some vector function $F(t)$ and vector c_1. Substituting this into the last equation in (5.12) gives

$$2F''(t) - 2if(t)F'(t) + 2c^2F(t) + c(b+c)c_1 = 0.$$

Thus, we get $F(t) = z(t) - ((b+c)/2c)c_1$, where $z = z(t)$ is a $\boldsymbol{C}^3$-valued solution of

$$z''(t) - if(t)z'(t) + c^2z(t) = 0. \tag{5.16}$$

Consequently, we obtain

$$L = e^{ibs}\left\{\left(2z(t) - \left(\frac{b}{c}\right)c_1\right)\cos(cs) + ic_1\sin(cs)\right\}. \tag{5.17}$$

Since $|L|^2 = 1$, (5.17) implies that

$$|c_1| = 1, \qquad |2cz(t) - bc_1|^2 = c^2, \qquad \langle z(t), ic_1\rangle = 0. \tag{5.18}$$

It follows immediately from (5.17) that

$$L_s = \frac{ie^{ibs}}{c}\{(c_1 + 2bcz(t))\cos(cs) + 2ic^2z(t)\sin(cs)\},$$
$$L_t = 2\cos(cs)e^{ibs}z'(t). \tag{5.19}$$

Thus, by applying the first equation in (5.11) and (5.19), we also have

$$|z'(t)| = \frac{1}{2}, \qquad |c_1 + 2bcz(t)|^2 = c^2, \qquad |z| = \frac{1}{2c}. \tag{5.20}$$

From $|c_1| = 1$, $|z| = 1/2c$ and the second equation in (5.20) we get $\langle z(t), c_1\rangle$=0. Also, by applying Lemma 2, we know that $z = z(t)$ is a Legendre curve of speed 1/2 in $S^3(1/2c) \subset \boldsymbol{C}^2 \subset \boldsymbol{C}^3$. So, if we choose $c_1 = (1, 0, 0)$, we obtain from (5.17) that

$$L = e^{ibs}\left(i\,\sin(cs) - \frac{b}{c}\,\cos(cs), 2z_1(t)\,\cos(cs), 2z_2(t)\,\cos(cs)\right)$$

Consequently, restricted to U_1, the Lagrangian surface in $CP^2(4)$ is congruent to the composition $\pi \circ L$, where L is given by Case (18).

Next, assume that θ is a non-constant function on an open interval $I \ni 0$. From (5.14) we find

$$L = e^{i(b-c)s}A(t) + e^{i(b+c)s}\int_0^t A'(t)e^{2i\theta}\,\mathrm{d}t + K(s), \qquad c = \sqrt{b^2+1} \tag{5.21}$$

for some $\boldsymbol{C}^3$-valued function K. Substituting this into the first equation in (5.12) gives $K'' = 2ibK' - K$. Solving this equation gives $K = a_1e^{i(b-c)s} + a_2e^{i(b+c)s}$ for some vectors $a_1, a_2 \in \boldsymbol{C}^3$. Hence, we obtain

$$L = e^{i(b-c)s}z(t) + e^{i(b+c)s}w(t), \tag{5.22}$$

where $z(t) = A(t) + a_1$ and $w(t) = \int_0^t A'(t)e^{2i\theta}\,\mathrm{d}t + a_2$.

Since $|L| = 1$, (5.22) implies that $1 = |A(t)|^2 + |W(t)|^2 + 2\langle ze^{2ics}, w\rangle$. Hence, by applying $\langle z, e^{2ics}w\rangle = \cos(2cs)\langle z, w\rangle + \sin(2cs)\langle z, iw\rangle$, we find

$$\langle z, w\rangle = \langle z, iw\rangle = 0, \qquad |z(t)|^2 + |w(t)|^2 = 1.$$

Also, from (5.22), we have

$$\begin{aligned} \tilde{L}_s &= i(b-c)e^{i(b-c)s}z(t) + i(b+c)e^{i(b+c)s}w(t),\\ L_t &= z'(t)e^{i(b-c)s}(1 + e^{2i(cs+\theta(t))}). \end{aligned} \tag{5.23}$$

Applying these gives $|z'(t)| = |w'(t)| = 1/2$, $|z(t)|^2 = b + c/2c$, and $|w(t)|^2 = c - b/2c$. So, after differentiating the last equation, we have $\langle z'e^{2i\theta}, w\rangle = 0$. Moreover, by applying $\langle L, L_t\rangle = \langle L_s, iL_t\rangle = 0$, we get

$$\langle z, e^{2i(cs+\theta)}z'\rangle + \langle z', e^{2ics}w\rangle = (b-c)\langle z, e^{2i(cs+\theta)}z'\rangle + (b+c)\langle z', e^{2ics}w\rangle = 0.$$

Therefore, we obtain $\langle A, e^{2i(cs+\theta)}z'\rangle = \langle z', e^{2ics}w\rangle = 0$ which implies that

$$\langle z, iz'\rangle = \langle z', w\rangle = \langle z', iw\rangle = \langle w', iw\rangle = 0.$$

Thus, $z : I \to S^5(\sqrt{b+c}/\sqrt{2c}) \subset \boldsymbol{C}^3$ and $w : I \to S^5(\sqrt{c-b}/\sqrt{2c}) \subset \boldsymbol{C}^3$ are Legendre curves of constant speed 1/2.

Now, by substituting (5.22) into the last equation in (5.13), we find

$$z''(t) = i(H(t) - \theta'(t))z'(t) - \frac{c(c-b)}{2}z(t) - \frac{c(b+c)}{2}e^{-2i\theta}w(t). \tag{5.24}$$

Since $w'(t) = e^{2i\theta}z'(t)$, $w = w(t)$ is a parallel normal vector field. Consequently, $z : I \to S^5(\sqrt{b+c}/\sqrt{2c}) \subset \boldsymbol{C}^3$ is a special Legendre curve of constant speed 1/2 and $w : I \to S^5(\sqrt{c-b}/\sqrt{2c}) \subset \boldsymbol{C}^3$ is an associated special Legendre curve of z with the same constant speed. Therefore, the Lagrangian surface is congruent to the composition $\pi \circ L$, where L is given by Case (5).

Case (I.ii). $\lambda \neq 2\mu$ *on an open subset* $U_2 \subset V_1$. In this case, (5.2), (5.3) and $\nabla_{e_1} e_1 = 0$ imply $e_2\lambda = e_2\mu = 0$. Thus, we obtain from (5.4) that

$$\omega_1^2(e_2) = \frac{\mu'(s)}{\lambda - 2\mu}. \tag{5.25}$$

If $\mu = 0$ identically on an open subset V of U_2, then (5.3) and (5.25) imply $K = 1$ and $\omega_1^2 = 0$ on V which is impossible. So, $\mu \neq 0$ almost everywhere on U_2.

Case (I.ii.a). $\lambda = \mu \neq 0$ *on* U_2. From (5.3) and (5.4) we get

$$K = 1, \quad e_1(\ln \mu) = -\omega_1^2(e_2), \quad e_2\lambda = e_2\mu = 0, \quad e_1\varphi = -\varphi\omega_1^2(e_2). \tag{5.26}$$

So, λ and μ depend only on s according to (5.8). Combining (5.7) and the second equation in (5.26) gives $G = F(u)/\mu(s)$. Hence, (5.6) reduces to

$$g = \mathrm{d}s \otimes \mathrm{d}s + \frac{\mathrm{d}t \otimes \mathrm{d}t}{\mu^2(s)}, \tag{5.27}$$

where $t = t(u)$ is an antiderivative of $F(u)$. Thus, (5.10) yields $\mu^{-1} = a\cos(s+b)$ with $a \neq 0$. Hence, after making a suitable translation in s, we obtain

$$g = \mathrm{d}s \otimes \mathrm{d}s + a^2\cos^2 s\,\mathrm{d}t \otimes \mathrm{d}t, \qquad \lambda = \mu = \frac{\sec s}{a}. \tag{5.28}$$

Without loss of generality, we may choose $a > 0$.

From (5.28) we find $\omega_1^2(e_2) = -\tan s$. Thus, we may obtain from the last equation in (5.26) that $\varphi_s = \varphi\tan s$ which gives $\varphi = f(t)\sec s$ for some function f.

From (5.1), (5.28) and the formula of Gauss, we obtain

$$\begin{aligned} L_{ss} &= \tfrac{i\sec s}{a} L_s - L, \\ L_{st} &= \left(\tfrac{i\sec s}{a} - \tan s\right) L_t, \\ L_{tt} &= (ia\cos s + a^2\sin s\cos s)L_s + iaf(t)L_t - a^2\cos^2 s L. \end{aligned} \tag{5.29}$$

Solving the first equation in (5.29) gives

$$L = z(t)(\sec s + \tan s)^{i/a}\cos s + C(t)\sqrt{1 + a^2\sin^2 s}\, e^{-i\tan^{-1}(a\sin s)}$$

for some $\boldsymbol{C}^3$-valued functions z and C. Substituting this into the second equation in (5.29) gives $C' = 0$. So, C is a constant vector, say c_1. Hence, L is given by

$$L = z(t)(\sec s + \tan s)^{i/a}\cos s + c_1\sqrt{1 + a^2\sin^2 s}\, e^{-i\tan^{-1}(a\sin s)}. \tag{5.30}$$

Substituting this into the last equation of (5.29) yields

$$z''(t) = iaf(t)z'(t) - (1 + a^2)z(t). \tag{5.31}$$

It follows immediately from (5.30) that

$$L_s = \frac{(a\,\sin s - i)\{c_1 a^2 e^{-i\,\tan^{-1}(a\,\sin s)}\,\cos s - z(t)\sqrt{1+a^2\,\sin^2 s}(\sec s + \tan s)^{i/a}\}}{a\sqrt{1+a^2\,\sin^2 s}},$$
$$L_t = z'(t)(\sec s + \tan s)^{i/a}\,\cos s. \tag{5.32}$$

Since $|L| = 1$, (5.30) and (5.32) imply that

$$|z(t)| = \frac{a}{\sqrt{1+a^2}}, \qquad |c_1| = \frac{1}{\sqrt{1+a^2}}, \qquad |z'(t)| = a, \qquad \langle c_1, z\rangle = 0.$$

Moreover, from $\langle L_s, iL_t\rangle = 0$, we also have $\langle c_1, iz(t)\rangle = \langle z(t), iz'(t)\rangle = 0$. These shows that $z = z(t)$ is a Legendre curve in $S^3(a/\sqrt{1+a^2}) \subset \boldsymbol{C}^2$ with constant speed a, where $\boldsymbol{C}^2$ is a complex hyperplane with c_1 as its hyperplane normal. Consequently, the Lagrangian surface, restricted to U_2, is obtained from Case (19).

Case (I.ii.b). $\lambda \neq \mu,\ \mu \neq 0$ *on an open subset* $W_1 \subset U_2$. In this case, (5.3) implies $K \neq 1$. Moreover, from (5.3), (5.5) and (5.7), we have

$$\lambda = \mu + \tfrac{K-1}{\mu},$$
$$\omega_2^1(e_2) = e_1(\ln\sqrt{|K-\mu^2-1|}) = e_1(\ln\varphi) = -e_1(\ln G) \tag{5.33}$$

on W_1, where G is defined by (5.6). Hence, we get

$$G\sqrt{|K-\mu^2-1|} = p(t), \qquad \varphi G = f(t), \tag{5.34}$$

for some positive real-valued function p and and non-zero real-valued function f.

We divide this case into several cases.

Case (I.ii.b.1). $K = b^2 > \mu^2 + 1 > 1$ *on a neighborhood* $W_{1,1}$ *of a point* $p \in W_1$. Without loss of generality, we may choose $b > 1$. From (5.10) and (5.34) we get

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t,$$
$$\mu^2 = b^2 - 1 - p^2(t)\,\sec^2(bs + \theta(t)), \qquad \varphi = f(t)\,\sec(bs + \theta(t)). \tag{5.35}$$

It follows from (5.5) that $\mu = \mu(s)$. Thus, by differentiating the second equation in (5.35), we get $(\ln p(t))' = \partial(\ln\cos(bs + \theta(t)))/\partial t$. Hence, $p(t) = k(s)\cos(bs + \theta(t))$ for some function $k(s)$. Differentiating the last equation with respect to s gives $(\ln k(s))' = b\,\tan(bs + \theta(t))$. Therefore, θ and p are constant. We may assume $\theta = 0$ by applying a suitable translation in s. Hence, we obtain from (5.35) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + (\cos^2 bs)\,\mathrm{d}t \otimes \mathrm{d}t,$$
$$\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = -b\,\tan(bs)\frac{\partial}{\partial t}, \qquad \nabla_{\partial/\partial t}\frac{\partial}{\partial t} = \frac{b}{2}\,\sin(2bs)\frac{\partial}{\partial s}, \tag{5.36}$$

$$\lambda = \frac{2a^2 - c^2\,\sec^2 bs}{\sqrt{a^2 - c^2\,\sec^2 bs}}, \qquad \mu = \sqrt{a^2 - c^2\,\sec^2 bs}, \qquad \varphi = f(t)\,\sec bs, \tag{5.37}$$

where $c = p$ is a positive number and $a = \sqrt{b^2 - 1}$.

From (5.1), (5.36), (5.37) and the formula of Gauss, we get

$$L_{ss} = i\frac{2a^2 - c^2 \sec^2 bs}{\sqrt{a^2 - c^2 \sec^2 bs}} L_s - L,$$
$$L_{st} = (i\sqrt{a^2 - c^2 \sec^2 bs} - b \tan bs)L_t, \tag{5.38}$$
$$L_{tt} = (b \sin bs + i\sqrt{a^2 \cos^2 bs - c^2}) \cos bs L_s + if(t)L_t - \cos^2 bs L.$$

After solving the second equation in (5.38) we find

$$L = B(\cos bs) \exp i \left\{ \frac{a}{b} \sin^{-1}\left(\frac{a \sin bs}{\sqrt{a^2 - c^2}}\right) - \frac{c}{b} \tan^{-1}\left(\frac{c \tan bs}{\sqrt{a^2 - c^2 \sec^2 bs}}\right)\right\} + A$$

for some $\boldsymbol{C}^3$-valued functions $A = A(s)$ and $B = B(t)$. Substituting this into the first equation in (5.38) yields

$$A''(s) = i\frac{2a^2 - c^2 \sec^2 bs}{\sqrt{a^2 - c^2 \sec^2 bs}} A'(s) - A(s). \tag{5.39}$$

Solving (5.39) gives

$$A(s) = c_0 \cos(bs) \exp i \left\{ \frac{a}{b} \sin^{-1}\left(\frac{a \sin bs}{\sqrt{a^2 - c^2}}\right) - \frac{c}{b} \tan^{-1}\left(\frac{c \tan bs}{\sqrt{a^2 - c^2 \sec^2 bs}}\right)\right\}$$
$$+ c_1 \frac{\sqrt{b^2 - c^2 - \cos^2 bs}(\sqrt{a^2 \cos^2 bs - c^2} + ia \sin bs)^{a/b}}{\exp(i \tan^{-1}(b \sin bs/\sqrt{a^2 \cos^2 bs - c^2}))}$$

for some vectors $c_0, c_1 \in \boldsymbol{C}^3$. Hence we obtain

$$L = z(t) \cos(bs) \exp i \left\{ \frac{a}{b} \sin^{-1}\left(\frac{a \sin bs}{\sqrt{a^2 - c^2}}\right) - \frac{c}{b} \tan^{-1}\left(\frac{c \tan bs}{\sqrt{a^2 - c^2 \sec^2 bs}}\right)\right\}$$
$$+ c_1 \frac{\sqrt{b^2 - c^2 - \cos^2 bs}(\sqrt{a^2 \cos^2 bs - c^2} + ia \sin bs)^{a/b}}{\exp(i \tan^{-1}(b \sin bs/\sqrt{a^2 \cos^2 bs - c^2}))}, \tag{5.40}$$

where $z(t) = A(t) + c_0$. Since $|L| = 1$, (5.40) implies

$$|z|^2 = \frac{1}{b^2 - c^2}, \qquad |c_1|^2 = \frac{1}{(b^2 - c^2)(a^2 - c^2)^{2a/b}}, \qquad \langle z, c_1 \rangle = 0. \tag{5.41}$$

Also, from (5.36), (5.39) and $\langle L_s, iL_t \rangle = 0$, we find $|z'(t)| = 1$ and $\langle z, ic_1 \rangle = 0$. Substituting these into the last equation of (5.38) yields

$$z''(t) = if(t)z'(t) - (b^2 - c^2)z(t).$$

Thus, $z(t)$ is a unit speed Legendre curve in $S^3(1/\sqrt{b^2 - c^2}) \subset \boldsymbol{C}^2$, where $\boldsymbol{C}^2$ is a complex hyperplane with hyperplane normal c_1. Consequently, the Lagrangian surface, restricted to $W_{1,1}$, is congruent to $\pi \circ L$ with L given by Case (20).

Case (I.ii.b.2). $0 < K = b^2 < \mu^2 + 1$ *on a neighborhood* $W_{1,2}$ *of a point* $p \in W_1$. Without loss of generality, we may assume $b > 0$. From (5.10) and (5.34) we get

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2(bs + \theta(t))\,\mathrm{d}t \otimes \mathrm{d}t,$$
$$\mu^2 = b^2 - 1 + p^2(t)\,\sec^2(bs + \theta(t)), \qquad \varphi = f(t)\,\sec(bs + \theta(t)). \tag{5.42}$$

Since $\mu = \mu(s)$ and $p(t)\,\sec(bs + \theta(t))$ depend only on s according to the second equation in (5.42), $p(t)$ and $\theta(t)$ both are constant as in Case (I.ii.c.1). So, we have $\theta = 0$ after applying a suitable translation in s. Hence, we obtain from (5.42) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + \cos^2 bs\,\mathrm{d}t \otimes \mathrm{d}t,$$
$$\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = -b\,\tan bs\frac{\partial}{\partial t}, \qquad \nabla_{\partial/\partial t}\frac{\partial}{\partial t} = \frac{b}{2}\,\sin 2bs\frac{\partial}{\partial s} \tag{5.43}$$

and

$$\lambda = \frac{2b^2 - 2 + c^2\,\sec^2 bs}{\sqrt{b^2 - 1 + c^2\,\sec^2 bs}}, \qquad \mu = \sqrt{b^2 - 1 + c^2\,\sec^2 bs},$$
$$\varphi = f(t)\,\sec bs, \tag{5.44}$$

where $c = p$ is a positive number. From (5.1), (5.43) (5.44) and the formula of Gauss, we have

$$L_{ss} = i\frac{2b^2 - 2 + c^2\,\sec^2 bs}{\sqrt{b^2 - 1 + c^2\,\sec^2 bs}}L_s - L,$$
$$L_{st} = (i\sqrt{b^2 - 1 + c^2\,\sec^2 bs} - b\,\tan bs)L_t, \tag{5.45}$$
$$L_{tt} = (b\,\sin bs + i\sqrt{c^2 + (b^2 - 1)\,\cos^2 bs})\,\cos bs L_s + ifL_t - \cos^2 bs\,L.$$

Case (I.ii.b.2.1). $b > 1$. In this case, (5.45) becomes

$$L_{ss} = i\frac{2a^2 + c^2\,\sec^2(bs)}{\sqrt{a^2 + c^2\,\sec^2(bs)}}L_s - L,$$
$$L_{st} = (i\sqrt{a^2 + c^2\,\sec^2(bs)} - b\,\tan(bs))L_t, \tag{5.46}$$
$$L_{tt} = (b\,\sin bs + i\sqrt{c^2 + a^2\,\cos^2 bs})(\cos bs)L_s + ifL_t - (\cos^2 bs)L$$

with $a = \sqrt{b^2 - 1}$. After solving the first equation in (5.46) we find

$$L = C(t)\frac{\sqrt{b^2 + c^2 - \cos^2 bs}(\sqrt{a^2\,\cos^2 bs + c^2} + ia\,\sin bs)^{a/b}}{\exp(i\,\tan^{-1}(b\,\sin bs/\sqrt{a^2\,\cos^2 bs + c^2}))}$$
$$+ z(t)(\cos bs)(\sqrt{a^2\,\cos^2 bs + c^2} + ia\,\sin bs)^{a/b}$$
$$\times\,\exp\left\{\frac{ic}{b}\,\tanh^{-1}\left(\frac{c\,\tan bs}{\sqrt{a^2 + c^2\,\sec^2 bs}}\right)\right\} \tag{5.47}$$

for some $\boldsymbol{C}^3$-valued functions z and C. Substituting this into the second equation in (5.45) shows that C is a constant vector, say $c_1 \in \boldsymbol{C}^3$. Since $|L| = 1$, (5.47) implies

$$|z|^2 = |c_1|^2 = \frac{1}{(b^2+c^2)(a^2+c^2)^{a/b}}, \qquad \langle z, c_1\rangle = 0. \tag{5.48}$$

Also, from (5.43), (5.47) and $\langle L_s, iL_t\rangle = 0$, we find

$$|z'(t)|^2 = (a^2+c^2)^{-a/b}, \qquad \langle z, ic_1\rangle = 0. \tag{5.49}$$

Also, by substituting (5.47) into the last equation of (5.46), we get

$$z''(t) = if(t)z'(t) - (b^2+c^2)z(t). \tag{5.50}$$

Therefore, z is a Legendre curve in $S^3(1/\sqrt{b^2+c^2}) \subset \boldsymbol{C}^2$ with speed $1/(a^2+c^2)^{a/2b}$, where $\boldsymbol{C}^2$ is a complex hyperplane with c_1 as its normal. Hence the Lagrangian surface, restricted to $W_{1,1}$, is congruent to $\pi \circ L$ with L given by Case (21).

Case (I.ii.b.2.2). $b = 1$. In this case, (5.44) becomes

$$L_{ss} = ic\ \sec s\, L_s - L, \qquad L_{st} = (ic\ \sec s - \tan s)L_t,$$
$$L_{tt} = (\sin s + ic)(\cos s)L_s + ifL_t - (\cos^2 s)L. \tag{5.51}$$

After solving the first equation in (5.51) we find

$$L = z(t)(\cos s)e^{2ic\ \tanh^{-1}(\tan s/2)} + C(t)\sqrt{c^2+\sin^2 s}\ e^{-i\ \tan^{-1}(c^{-1}\ \sin s)} \tag{5.52}$$

for some $\boldsymbol{C}^3$-valued functions z and C. Substituting this into the second equation in (5.45) shows that C is a constant vector, say c_1. Since $|L| = 1$, (5.52) implies

$$|z|^2 = |c_1|^2 = \frac{1}{1+c^2}, \qquad \langle z, c_1\rangle = 0. \tag{5.53}$$

From (5.43), (5.52) and $\langle L_s, iL_t\rangle = 0$ we find $|z'(t)|^2 = 1$, $\langle z, ic_1\rangle = 0$. Also, by substituting (5.52) into the last equation of (5.51), we get

$$z''(t) = if(t)z'(t) - (1+c^2)z(t). \tag{5.54}$$

Thus, z is a unit speed Legendre curve in $S^3(1/\sqrt{1+c^2}) \subset \boldsymbol{C}^2$, where $\boldsymbol{C}^2$ is a complex hyperplane with c_1 as its hyperplane normal. Consequently, the Lagrangian surface, restricted to $W_{1,1}$, is congruent to $\pi \circ L$ with L given by Case (22).

Case (I.ii.b.2.3). $0 < b < 1$. In this case, (5.44) becomes

$$L_{ss} = i\frac{c^2\ \sec^2 bs - 2a^2}{\sqrt{c^2\ \sec^2 bs - a^2}}L_s - L, \qquad a = \sqrt{1-b^2},$$
$$L_{st} = (i\sqrt{c^2\ \sec^2 bs - a^2} - b\ \tan bs)L_t, \tag{5.55}$$
$$L_{tt} = (b \sin bs + i\sqrt{c^2 - a^2\ \cos^2 bs})\ \cos bsL_s + ifL_t - \cos^2 bsL.$$

After solving the first and second equations in (5.55) we find

$$L = z(t)\frac{(\cos bs)\ \exp i(b^{-1}c\ \tanh^{-1}(c\ \sin bs/\sqrt{c^2 - a^2\ \cos^2 bs}))}{\exp i(ab^{-1}\ \sinh^{-1}(a\ \sin bs/\sqrt{c^2 - a^2}))}$$
$$+ \frac{c_1\sqrt{b^2 + c^2 - \cos^2 bs}\exp i\{(a^2(2b^2 + 2c^2 - 1)/2b^2(a^2 - c^2)) \times \tan^{-1}(\sqrt{c^2 - a^2\ \cos^2 bs}/b\ \sin bs)\}}{(a\ \sin bs + \sqrt{c^2 - a^2\ \cos^2 bs})^{ia/b}\ \exp i\{(a^2 - 2c^2/2b^2(a^2 - c^2)) \times \tan^{-1}(b\ \sin bs/\sqrt{c^2 - a^2\ \cos^2 bs})\}} \quad (5.56)$$

for some $\boldsymbol{C}^3$-valued function $z = z(t)$ and constant vector c_1.

Since $|L| = 1$, (5.56) implies that

$$|z|^2 = |c_1|^2 = \frac{1}{(b^2 + c^2)}, \qquad \langle z, c_1\rangle = 0. \quad (5.57)$$

From (5.43), (5.56) and $\langle L_s, iL_t\rangle = 0$ we find $|z'(t)| = 1$ and $\langle z, ic_1\rangle = 0$. Also, substituting (5.56) into the last equation of (5.55) yields

$$z''(t) = if(t)z'(t) - (b^2 + c^2)z(t).$$

Thus, z is a unit speed Legendre curve in $S^3(1/\sqrt{b^2 + c^2}) \subset \boldsymbol{C}^2$, where $\boldsymbol{C}^2$ is a complex hyperplane with c_1 as its hyperplane normal. Consequently, the Lagrangian surface, restricted to $W_{1,1}$, is congruent to $\pi \circ L$ with L given by Case (23).

Case (I.ii.b.3). *$K = 0$ on a neighborhood $W_{1,3}$ of a point $p \in W_1$.* In this case, we obtain from (5.10) and (5.34) that

$$g = \mathrm{d}s \otimes \mathrm{d}s + (cs + \theta(t))^2\, \mathrm{d}t \otimes \mathrm{d}t,$$
$$\mu^2 = \frac{p^2(t)}{(cs + \theta(t))^2} - 1, \qquad \varphi = \frac{A(t)}{cs + \theta(t)}, \quad (5.58)$$

If $c = 0$, we get $g = \mathrm{d}s \otimes \mathrm{d}s + \theta^2(t)\,\mathrm{d}t \otimes \mathrm{d}t$ and $\varphi = \varphi(t)$. Thus, we have

$$g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{d}v \otimes \mathrm{d}v, \qquad \varphi = \varphi(v), \quad (5.59)$$

where $v = v(t)$ is an antiderivative of $\theta(t)$. Since $\lambda = \lambda(s)$ and $\mu = \mu(s)$ for Case (I.ii), we know from (5.33) that μ is constant.

If $\mu = 0$, then (5.1) and the equation of Gauss imply that $K = 1$ which is a contradiction. Hence μ is a non-zero constant, say a. Thus, we obtain from (5.3) that $\lambda = a - a^{-1}$. Therefore, (5.1), (5.59) and the formula of Gauss imply that

$$L_{ss} = i\left(a - \frac{1}{a}\right)L_s - L, \qquad L_{sv} = iaL_v, \qquad L_{vv} = iaL_s + i\varphi(v)L_v - L. \quad (5.60)$$

Solving the first and the second equations in (5.60) gives

$$L = z(v)e^{ias} + c_1e^{-is/a} \quad (5.61)$$

for some $\boldsymbol{C}^3$-valued function $z = z(v)$ and vector $c_1 \in \boldsymbol{C}^3$. Since $|L| = |L_s| = |L_v| = 1$ and $\langle L_s, iL_y\rangle = 0$, we derive from (5.61) that

$$|z|^2 = \frac{1}{1+a^2}, \qquad |z'|^2 = 1, \qquad |c_1|^2 = \frac{a^2}{1+a^2}, \qquad \langle z, c_1\rangle = \langle z, ic_1\rangle = 0. \tag{5.62}$$

Also, by substituting (5.61) into the last equation in (5.60), we find

$$z''(y) = i\varphi(y)z'(y) - (1+a^2)z(y). \tag{5.63}$$

Consequently, the surface is congruent to $\pi \circ L$ where L is given by Case (16).

Next, assume that $c \neq 0$. In this case, $p(t)$ and $\theta(t)$ are constant due to $\mu = \mu(s)$ and the second equation in (5.58). So, we have $\theta = 0$ after applying a suitable translation in s. Consequently, we have $g = \mathrm{d}s \otimes \mathrm{d}s + c^2s^2\, \mathrm{d}t \otimes \mathrm{d}t$. So, if we put $v = ct$ and $p = b$, we find

$$g = \mathrm{d}s \otimes \mathrm{d}s + s^2\, \mathrm{d}v \otimes \mathrm{d}v, \tag{5.64}$$

$$\lambda = \frac{b^2 - 2s^2}{s\sqrt{b^2 - s^2}}, \qquad \mu = \frac{\sqrt{b^2 - s^2}}{s}, \qquad \varphi = \frac{f(v)}{s}, \tag{5.65}$$

From (5.58), (5.64) and the formula of Gauss, we obtain

$$\begin{aligned} &L_{ss} = i\frac{b^2 - 2s^2}{s\sqrt{b^2 - s^2}}L_s - L, \qquad L_{sv} = \left(\frac{1}{s} + i\frac{\sqrt{b^2 - s^2}}{s}\right)L_v, \\ &L_{vv} = (is\sqrt{b^2 - s^2} - s)L_s + if(v)L_v - s^2L. \end{aligned} \tag{5.66}$$

After solving the first and second equations in (5.66) we obtain

$$L = e^{i\sqrt{b^2-s^2}}\left\{\frac{z(v)s^{1+ib}}{(c + \sqrt{b^2 - s^2})^{ib}} + \frac{c_1\sqrt{1 + b^2 - s^2}}{e^{i\,\tan^{-1}\sqrt{b^2-s^2}}}\right\} \tag{5.67}$$

for some $\boldsymbol{C}^3$-valued functions z and constant vector $c_1 \in \boldsymbol{C}^3$. Since $|L| = 1$, (5.67) implies

$$|z|^2 = |c_1|^2 = \frac{1}{1+b^2}, \qquad \langle z, c_1\rangle = 0. \tag{5.68}$$

Also, from (5.43), (5.56) and $\langle L_s, iL_v\rangle = 0$ we find $|z'(v)| = 1$ and $\langle z, ic_1\rangle = 0$. Moreover, by substituting (5.67) into the last equation of (5.66), we get

$$z''(v) = if(v)z'(v) - (1+b^2)z(v).$$

Therefore, z is a unit speed Legendre curve in $S^3(\frac{1}{\sqrt{1+b^2}}) \subset \boldsymbol{C}^2$, where $\boldsymbol{C}^2$ is a complex hyperplane with c_1 as its hyperplane normal. Consequently, the Lagrangian surface, restricted to $W_{1,1}$, is congruent to $\pi \circ L$ with L given by Case (17).

Case (I.ii.b.4). *$K = -b^2 < 0$ on a neighborhood $W_{1,4}$ of a point $p \in W_1$*. We may assume $b > 0$. From (5.10) and (5.34) we get

$$\begin{aligned} &g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2(bs + \theta(t))\, \mathrm{d}t \otimes \mathrm{d}t, \\ &\mu^2 = p^2(t)\,\mathrm{sech}^2(bs + \theta(t)) - b^2 - 1, \qquad \varphi = f(t)\,\mathrm{sech}\,(bs + \theta(t)), \end{aligned} \tag{5.69}$$

for some function $p(t)$ and $f(t)$ satisfying $p^2 > b^2 + 1$.

Since $\mu = \mu(s)$, the second equation in (5.69) implies that $p(t)\mathrm{sech}\,(bs + \theta(t))$ depends on only on s which is impossible unless both p and θ are constant. Thus, we have $\theta = 0$ after applying a suitable translation in s. If we denote the constant p by c, we obtain $c^2 \geq b^2 + 1$. Moreover, we have

$$\begin{aligned} &g = \mathrm{d}s \otimes \mathrm{d}s + \cosh^2 bs\, \mathrm{d}t \otimes \mathrm{d}t, \qquad \varphi = f(t)\,\mathrm{sech}\, bs, \\ &\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = b\tanh bs\frac{\partial}{\partial t}, \qquad \nabla_{\partial/\partial t}\frac{\partial}{\partial t} = -\frac{b}{2}\,\sinh 2bs\frac{\partial}{\partial s} \\ &\lambda = \frac{c^2\,\mathrm{sech}^2 bs - 2a^2}{\sqrt{c^2\,\mathrm{sech}^2 bs - a^2}}, \qquad \mu = \sqrt{c^2\,\mathrm{sech}^2 bs - a^2}, \qquad a = \sqrt{b^2 + 1}. \end{aligned} \tag{5.70}$$

From (5.1), (5.70) and the formula of Gauss, we find

$$\begin{aligned} &L_{ss} = i\frac{c^2\,\mathrm{sech}^2 bs - 2a^2}{\sqrt{c^2\,\mathrm{sech}^2 bs - a^2}}L_s - L, \qquad a = \sqrt{b^2 + 1}, \\ &L_{st} = (i\sqrt{c^2\,\mathrm{sech}^2 bs - a^2} + b\tanh bs)L_t, \\ &L_{tt} = (i\sqrt{c^2 - a^2\,\cosh^2 bs} - b\,\sinh bs)\,\cosh bsL_s + ifL_t - \cosh^2 bsL. \end{aligned} \tag{5.71}$$

Solving the first and the second equations of this system gives

$$\begin{aligned} L = {}& e^{-iab^{-1}\,\sin^{-1}((a\,\sinh bs)/\sqrt{c^2-a^2})}\{c_1(i\sqrt{c^2 - a^2\,\cosh^2 bs} - b\,\sinh bs) \\ &+ z(t)e^{ib^{-1}c\tan^{-1}((c\,\sinh bs)/\sqrt{c^2-a^2\,\cosh^2 bs})}\,\cosh bs\} \end{aligned} \tag{5.72}$$

for some vector c_1 and vector function z. Since $|L| = |L_s| = 1, |L_t| = \cosh bs$ and $\langle L_s, iL_y\rangle = 0$, we obtain from (5.73) that

$$|z|^2 = |c_1|^2 = \frac{1}{c^2 - b^2}, \qquad |z'|^2 = 1, \qquad \langle z, c_1\rangle = \langle z, ic_1\rangle = 0.$$

By substituting (5.72) into the third equation of (5.71) we obtain

$$z''(t) = if(t)z'(t) - (c^2 - b^2)z(t).$$

Thus, $z(t)$ is a unit speed Legendre curve in $S^3(1/\sqrt{c^2 - b^2}) \subset \boldsymbol{C}^2$. Therefore, the immersion L, restricted $W_{1,4}$, is congruent to $\pi \circ L$, where L is given by case (24).

Case (II). *$\nabla_{e_1}e_1 \neq 0$ on an open subset $V_2 \subset V$*. In this case, $\omega_1^2(e_1)$ is never zero on V_2. Since Span $\{e_1\}$ and Span $\{e_2\}$ are of rank one, there exists local coordinates $\{x, y\}$ on V_2 such that $\partial/\partial x$, $\partial/\partial y$ are parallel to e_1, e_2, respectively. Thus, the metric tensor g takes the form:

$$g = E^2\,\mathrm{d}x \otimes \mathrm{d}x + G^2\,\mathrm{d}y \otimes \mathrm{d}y, \tag{5.73}$$

for some positive functions E, G. We may assume $\partial/\partial x = Ee_1$, $\partial/\partial y = Ge_2$. From (5.73) we have

$$\omega_2^1(e_1) = \frac{E_y}{EG}, \qquad \omega_1^2(e_2) = \frac{G_x}{EG}, \qquad E_y = \frac{\partial E}{\partial y}, \qquad G_x = \frac{\partial G}{\partial x}. \tag{5.74}$$

If $\lambda = 2\mu$, (5.3) reduces to $K = \mu^2 + 1$ which implies μ is constant. So, the first equation in (5.2) and $\omega_1^2(e_1) \neq 0$ give $\varphi = 0$ which contradicts to $\varphi \neq 0$. Hence, we have $\lambda \neq 2\mu$. Also, the second equation in (5.2) and $\omega_1^2(e_1) \neq 0$ give $e_2\lambda \neq 0$. So, λ is a non-trivial function.

Case (II.i). $\mu = 0$ *on* V_2. In this case, we get from (5.3) that $K = 1$. Moreover, from (5.2) we find

$$\varphi\omega_1^2(e_1) = \lambda\omega_2^1(e_2), \qquad e_2\lambda = \lambda\omega_1^2(e_1), \qquad e_1\varphi = \varphi\omega_2^1(e_2), \tag{5.75}$$

It follows from (5.74) and (5.75) that $\lambda E = \eta(x)$ and $\varphi G = k(y)$ for some functions $\eta(x)$ and $k(y)$. Hence, (5.73) becomes

$$g = \frac{\eta^2(x)}{\lambda^2}\,\mathrm{d}x \otimes \mathrm{d}x + \frac{k^2(y)}{\varphi^2}\,\mathrm{d}y \otimes \mathrm{d}y. \tag{5.76}$$

If $u = u(x)$ and $v = v(y)$ are antiderivatives of $\eta(x)$ and $k(y)$ respectively, then (5.1) and (5.76) reduce to

$$g = \lambda^{-2}\,\mathrm{d}u \otimes \mathrm{d}u + \varphi^{-2}\,\mathrm{d}v \otimes \mathrm{d}v, \tag{5.77}$$

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \qquad h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = J\frac{\partial}{\partial v}. \tag{5.78}$$

From (5.77) we have

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial u} = -\frac{\lambda_u}{\lambda}\frac{\partial}{\partial u} + \frac{\varphi^2\lambda_v}{\lambda^3}\frac{\partial}{\partial v}, \qquad \nabla_{\partial/\partial u}\frac{\partial}{\partial v} = -\frac{\lambda_v}{\lambda}\frac{\partial}{\partial u} - \frac{\varphi_u}{\varphi}\frac{\partial}{\partial v},$$
$$\nabla_{\partial/\partial v}\frac{\partial}{\partial v} = \frac{\lambda^2\varphi_u}{\varphi^3}\frac{\partial}{\partial u} - \frac{\varphi_v}{\varphi}\frac{\partial}{\partial v}. \tag{5.79}$$

By applying (5.77), (5.78), (5.79) and the formula of Gauss, we obtain

$$\begin{aligned} L_{uu} &= (i - (\ln\lambda)_u)L_u + (\ln\varphi)_u L_v - \tfrac{1}{\lambda^2}L, \\ L_{uv} &= -(\ln\lambda)_v L_u - (\ln\varphi)_u L_v, \\ L_{vv} &= (\ln\lambda)_v L_u + (i - (\ln\varphi)_v)L_v - \tfrac{1}{\varphi^2}L. \end{aligned} \tag{5.80}$$

By applying (5.75) and (5.77), we find

$$\lambda\omega_1^2(e_1) = \varphi\lambda_v, \qquad \varphi\omega_2^1(e_2) = \lambda\varphi_u, \qquad \varphi^3\lambda_v = \lambda^3\varphi_u. \tag{5.81}$$

Since $K = 1$, (5.77) and (5.81) imply that

$$\left(\frac{\varphi\lambda_v}{\lambda^2}\right)_v + \left(\frac{\lambda\varphi_u}{\varphi^2}\right)_u = \frac{1}{\lambda\varphi}. \tag{5.82}$$

If $\lambda_v = 0$, we get $\varphi_u = 0$ from (5.81) which contradicts (5.82). Hence, we must have $\lambda_v \neq 0$. Similarly, we also have $\varphi_u \neq 0$. So, the last equation in (5.81) gives

$$\frac{\varphi\lambda_v}{\lambda^2} = \frac{\lambda\varphi_u}{\varphi^2} = f(u, v) \tag{5.83}$$

for some non-zero function f. Also, f is non-constant according to (5.82) and (5.83).

We divide Case (II.i) into two cases.

Case (II.i.a). $\lambda = \varphi \neq 0$ *on a neighborhood* O_1 *of a point in* $W_{2,1}$. In this case, (5.81) reduces to $\lambda_u = \lambda_v$. Thus, $\lambda = \varphi$ is a function of $s := u + v$. So, (5.82) yields $2\lambda(s)\lambda''(s) - 2\lambda'^2(s) + 1 = 0$. After solving this differential equation and applying a suitable translation in s, we obtain

$$\lambda = \frac{\cosh bs}{\sqrt{2}b}, \tag{5.84}$$

where b is a non-zero real number. Hence, system (5.80) reduces to

$$\begin{aligned} L_{uu} &= (i - b\tanh bs)L_u + b\tanh bsL_v - 2b^2\,\mathrm{sech}^2 bsL,\\ L_{uv} &= -b\tanh bs(L_u + L_v),\\ L_{vv} &= b\tanh bsL_u + (i - b\tanh bs)L_v - 2b^2\,\mathrm{sech}^2 bsL. \end{aligned} \tag{5.85}$$

If we put $t = u - v$ as well as $s = u + v$, then (5.85) gives

$$L_s = \tfrac{1}{2}(L_u + L_v), \qquad L_t = \tfrac{1}{2}(L_u - L_v), \qquad L_{ss} = \tfrac{1}{4}(L_{uu} + 2L_{uv} + L_{vv}),$$

$$L_{st} = \tfrac{1}{4}(L_{uu} - L_{vv}), \qquad L_{tt} = \tfrac{1}{4}(L_{uu} - 2L_{uv} + L_{vv}).$$

Thus, (5.85) becomes

$$L_{ss} = \left(\frac{i}{2} - b\tanh bs\right)L_s - b^2\,\mathrm{sech}^2 bsL, \qquad L_{st} = \left(\frac{i}{2} - b\tanh bs\right)L_t,$$

$$L_{tt} = \left(\frac{i}{2} + b\tanh bs\right)L_s - b^2\,\mathrm{sech}^2 bsL. \tag{5.86}$$

Solving the first two equations in (5.86) gives

$$L = \left(A(t)e^{is/2} + c_1\frac{\sqrt{(1+4b^2)\cosh^2 bs - 4b^2}}{e^{i\tan^{-1}(2b\tanh bs)}}\right)\mathrm{sech}bs \tag{5.87}$$

for some vector function A and vector c_1. Substituting (5.87) into the last equation of (5.86) yields $4A''(t) + (1 + 4b^2)A(t) = 0$. After solving this equation we find $A(t) = c_2\,\cos(\sqrt{1+4b^2}\,t/2) + c_3\,\sin(\sqrt{1+4b^2}\,t/2)$ for some $c_2, c_3 \in \boldsymbol{C}^3$. Hence, we obtain

$$\begin{aligned} L &= c_1 e^{-i\tan^{-1}(2b\tanh bs)}\sqrt{(1+4b^2)\,\cosh^2 bs - 4b^2}\,\mathrm{sech}bs\\ &\quad + e^{is/2}\,\mathrm{sech}bs\{c_2\,\cos(\sqrt{1+4b^2}\,t/2) + c_3\,\sin(\sqrt{1+4b^2}\,t/2)\} \end{aligned} \tag{5.88}$$

Thus, we obtain Case (25) after choosing suitable initial conditions.

Case (II.i.b). *$\varphi \neq 0$ and $\lambda \neq \varphi$ on a neighborhood O_2 of a point in $W_{2,1}$*. From (5.75) we know that $e_2\lambda, e_1\varphi, \omega_1^2(e_2)$ are nonzero on O_2. By applying (5.77) we get

$$g = \rho^2\, \mathrm{d}u \otimes \mathrm{d}u + \psi^2\, \mathrm{d}v \otimes \mathrm{d}v, \tag{5.89}$$

where $\rho = 1/\lambda$ and $\psi = 1/\varphi$. Since we have $\varphi_u, \lambda_v \neq 0$ in Case (II.i), we find $\rho_v, \psi_u \neq 0$. By applying (5.82) and (5.83) we find

$$\frac{\rho_v}{\psi} = \frac{\psi_u}{\rho} = -f, \qquad \left(\frac{\psi_u}{\rho}\right)_u + \left(\frac{\rho_v}{\psi}\right)_v + \rho\psi = 0. \tag{5.90}$$

The first equation in (5.90) implies $\rho_{uv} = f^2\rho - f_u\psi$. If $\rho = \rho(v)$, we obtain $f_u = f^2\rho/\psi = -f\rho\rho_v$ which implies $f(u,v) = H(v)e^{-u\rho\rho_v}$ and $\psi = -\rho_v e^{u\rho\rho_v}/H(v)$ for some function $H(v)$. Thus, we have $H^2(v) = \rho_v^2 e^{2u\rho\rho_v}$ by using $\psi_u = -f\rho$ which is impossible. Thus, we must also have $\rho_u \neq 0$. Hence, we obtain $\rho_u, \rho_v, \psi_u \neq 0$.

Next, assume that $\psi_v = 0$, i.e., $\psi = \psi(u)$. Then the first equation in (5.90) gives $\rho = \pm\sqrt{2\psi(u)\psi'(u)v + 2h(u)}$ for some function $h = h(u)$. By substituting this into the second equation in (5.90) and by applying the first equation, we get

$$2\psi'(\psi\psi' v + h) + \psi((\psi'^2 + \psi\psi'')v + h') + \psi^2\psi' + 4\psi(\psi\psi' v + h)^2 = 0.$$

Thus, ψ is constant which is a contradiction. Hence, we get $\rho_u, \rho_v, \psi_u, \psi_v \neq 0$.

From (5.89) and (5.90), we find

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial u} = \frac{\rho_u}{\rho}\frac{\partial}{\partial u} - \frac{\psi_u}{\psi}\frac{\partial}{\partial v}, \qquad \nabla_{\partial/\partial u}\frac{\partial}{\partial v} = \frac{\rho_v}{\rho}\frac{\partial}{\partial u} + \frac{\psi_u}{\psi}\frac{\partial}{\partial v},$$
$$\nabla_{\partial/\partial v}\frac{\partial}{\partial v} = -\frac{\rho_v}{\rho}\frac{\partial}{\partial u} + \frac{\psi_v}{\psi}\frac{\partial}{\partial v}. \tag{5.91}$$

Moreover, from (5.78), we have

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \qquad h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = J\frac{\partial}{\partial v}. \tag{5.92}$$

By combining (5.78), (5.89), (5.91), (5.92) and the formula of Gauss we obtain

$$L_{uu} = \left(i + \frac{\rho_u}{\rho}\right)L_u - \frac{\psi_u}{\psi}L_v - \rho^2 L, \qquad L_{uv} = \frac{\rho_v}{\rho}L_u + \frac{\psi_u}{\psi}L_v,$$
$$L_{vv} = -\frac{\rho_v}{\rho}L_u + \left(i + \frac{\psi_v}{\psi}\right)L_v - \psi^2 L. \tag{5.93}$$

A direct computation shows that the compatibility conditions: $L_{uuv} = L_{uvu}$ and $L_{uvv} = L_{vvu}$ hold if and only if (5.90) holds true. Thus, according to Proposition 1, the Lagrangian surface is locally given by Case (26).

Case (II.ii). *$\mu \neq 0$ and $\lambda \neq 2\mu$ on a neighborhood $V_{2,3}$ of a point $p \in V_2$.*

We divide this case into two cases: $\lambda = \mu$ or $\lambda \neq \mu$.

Case (II.ii.a). $\lambda = \mu$. Let θ is a solution of $\lambda(1 - 2\cos^2\theta) = \varphi\sin\theta\cos\theta$ and put $\hat{e}_1 = \cos\theta e_1 + \sin\theta e_2$, $\hat{e}_2 = -\sin\theta e_1 + \cos\theta e_2$, then (5.1) yields

$$h(\hat{e}_1, \hat{e}_1) = \hat{\lambda} J\hat{e}_1, \qquad h(\hat{e}_1, \hat{e}_1) = 0, \qquad h(\hat{e}_2, \hat{e}_2) = \hat{\varphi} J\hat{e}_2, \tag{5.94}$$

where $\hat{\lambda} = \sin^2\theta(2\lambda\cos\theta + \varphi\sin\theta) + \lambda\cos\theta$, $\hat{\varphi} = \cos^2\theta(\varphi\cos\theta - 2\lambda\sin\theta) - \lambda\sin\theta$. Hence, this case reduces to Case (I.ii.a) or Case (II.i) according to $\nabla_{e_1} e_1 = 0$ or $\nabla_{e_1} e_1 \neq 0$.

Case (II.ii.b). $\lambda \neq \mu$. The assumption $\nabla_{e_1} e_1 \neq 0$ for Case (II) and the second equation in (5.2) imply $e_2\lambda \neq 0$. Since $K = \lambda\mu - \mu^2 + 1$ is nonzero, we get

$$\mu e_j \lambda = (2\mu - \lambda) e_j \mu, \qquad j = 1, 2, \tag{5.95}$$

which implies $e_2\mu \neq 0$ as well. Combining (5.2) with (5.95) gives

$$\begin{aligned} &e_1\mu = \varphi\omega_1^2(e_1) + (\lambda - 2\mu)\omega_1^2(e_2), \\ &e_1\varphi = 4\mu\omega_2^1(e_1) + \varphi\omega_2^1(e_2), \qquad e_2(\ln\mu) = \omega_2^1(e_1). \end{aligned} \tag{5.96}$$

By applying the last equation of (5.96) and Cartan's structure equation, we find $\mathrm{d}(\mu^{-1}\omega^1) = 0$. Thus, there exists a function u such that

$$\mathrm{d}u = \frac{\omega^1}{\mu}, \qquad \frac{\partial}{\partial u} = \mu e_1. \tag{5.97}$$

Due to $K = \lambda\mu - \mu^2 + 1$, the first two equations in (5.96) give

$$4\mu e_1\mu + \varphi e_1\varphi = (4K - 4\mu^2 - 4 - \varphi^2)\omega_1^2(e_2). \tag{5.98}$$

Case (II.ii.b.1). $4K = 4\mu^2 + \varphi^2 + 4$. In this case, we have $K > \mu^2 + 1$. So, we may assume $\varphi = 2\sqrt{K - \mu^2 - 1}$. Thus, by $K = \lambda\mu - \mu^2 + 1$ and (5.96), we have

$$e_1\mu = \frac{K - \mu^2 - 1}{\mu}\omega_1^2(e_2) - 2\sqrt{K - \mu^2 - 1}\, e_2(\ln\mu). \tag{5.99}$$

Let $\Phi = \Phi(u, v)$ be a solution of

$$(\ln\Phi)_u = \frac{e_2\mu^2}{\sqrt{K - \mu^2 - 1}}. \tag{5.100}$$

Then, by $\partial/\partial u = \mu e_1$, (5.98), (5.100) and the last equation in (5.96), we obtain

$$\left[\frac{\partial}{\partial u}, \frac{\Phi e_2}{\sqrt{K - \mu^2 - 1}}\right] = \frac{\Phi}{\sqrt{K - \mu^2 - 1}}\left\{(\ln\Phi)_u + \frac{\mu^2 e_1\mu}{K - \mu^2 - 1} - \mu\omega_1^2(e_2)\right\}e_2 = 0.$$

Hence, there exists a coordinate system $\{u, v\}$ so that $\partial/\partial v = \Phi e_2/\sqrt{K - \mu^2 - 1}$. With respect to this coordinate system, we have

$$g = \mu^2\, \mathrm{d}u \otimes \mathrm{d}u + \frac{\Phi^2}{K - \mu^2 - 1}\, \mathrm{d}v \otimes \mathrm{d}v \tag{5.101}$$

$$\frac{\partial\Phi}{\partial u} = \frac{\partial\mu^2}{\partial v} \neq 0, \tag{5.102}$$

where the second equation is due to (5.100), (5.101) and $e_2\mu \neq 0$. Also, by applying (5.101) and (5.102), we know that the Gauss curvature K satisfies

$$\frac{-K\mu\Phi}{\sqrt{K-\mu^2-1}} = \left\{\left(\frac{2(K-\mu^2-1)\mu_v + \Phi\mu_u}{(K-\mu^2-1)^{3/2}}\right)_u + \left(\frac{\mu_v\sqrt{K-\mu^2-1}}{\Phi}\right)_v\right\}. \tag{5.103}$$

From (5.101) we find

$$\begin{aligned}
\nabla_{\partial/\partial u}\frac{\partial}{\partial u} &= \frac{\mu_u}{\mu}\frac{\partial}{\partial u} - \frac{(K-\mu^2-1)\mu\mu_v}{\Phi^2}\frac{\partial}{\partial v},\\
\nabla_{\partial/\partial u}\frac{\partial}{\partial v} &= \frac{\mu_v}{\mu}\frac{\partial}{\partial u} + \left(\frac{\mu\mu_u}{K-\mu^2-1} + \frac{\Phi_u}{\Phi}\right)\frac{\partial}{\partial v},\\
\nabla_{\partial/\partial v}\frac{\partial}{\partial v} &= -\frac{\Phi^2\mu\mu_u + (K-\mu^2-1)\Phi\Phi_u}{\mu^2(K-\mu^2-1)^2}\frac{\partial}{\partial u} + \left(\frac{\mu\mu_v}{K-\mu^2-1} + \frac{\Phi_v}{\Phi}\right)\frac{\partial}{\partial v}.
\end{aligned} \tag{5.104}$$

Thus, by (5.1), (5.101), (5.102) and the formula of Gauss, we obtain

$$\begin{aligned}
L_{uu} &= \left\{i(K+\mu^2-1) + \frac{\mu_u}{\mu}\right\}L_u - \frac{(K-\mu^2-1)\mu\mu_v}{\Phi^2}L_v - \mu^2 L,\\
L_{uv} &= \frac{\mu_v}{\mu}L_u + \mu\left\{i\mu + \frac{\mu_u}{K-\mu^2-1} + \frac{2\mu_v}{\Phi}\right\}L_v\\
L_{vv} &= \Phi\left\{\frac{i\Phi}{K-\mu^2-1} - \frac{\Phi\mu_u + 2(K-\mu^2-1)\mu_v}{\mu(K-\mu^2-1)^2}\right\}L_u\\
&\quad + \left\{2i\Phi + \frac{\mu\mu_v}{K-\mu^2-1} + \frac{\Phi_v}{\Phi}\right\}L_v - \frac{\Phi^2}{K-\mu^2-1}L.
\end{aligned} \tag{5.105}$$

A direct computation shows that the compatibility conditions: $L_{uuv} = L_{uvu}$ and $L_{uvv} = L_{vvu}$ hold if and only if (5.102) and (5.103) hold true. Therefore, the Lagrangian surface is locally given by Case (27).

Case (II.ii.b.2). $4K \neq 4\mu^2 + \varphi^2 + 4$. From (5.98) we get

$$\omega_1^2(e_2) = \frac{4\mu e_1\mu + \varphi e_1\varphi}{4(K-\mu^2-1)-\varphi^2}. \tag{5.106}$$

Thus, by applying (5.95), (5.96) and (5.106), we find

$$\begin{aligned}
&\omega_2^1(e_1) = e_2(\ln\mu), \qquad \omega_1^2(e_2) = e_1(\ln G),\\
&G = \frac{1}{\sqrt{|4(K-\mu^2-1)-\varphi^2|}}.
\end{aligned} \tag{5.107}$$

which implies $[\mu e_1, Ge_2] = 0$. Thus, there exists a coordinate system $\{u, v\}$ with $\partial/\partial u = \mu e_1$, $\partial/\partial v = Ge_2$. With respect to this coordinate system, we have

$$g = \mu^2\, \mathrm{d}u \otimes \mathrm{d}u + \frac{\mathrm{d}v \otimes \mathrm{d}v}{|4(K-\mu^2-1)-\varphi^2|}. \tag{5.108}$$

If $4(K-\mu^2-1) > \varphi^2$, then (5.108) implies

$$\begin{aligned}
\nabla_{\partial/\partial u}\frac{\partial}{\partial u} &= (\ln\mu)_u\frac{\partial}{\partial u} - \{4(K-\mu^2-1)-\varphi^2\}\mu\mu_v\frac{\partial}{\partial v},\\
\nabla_{\partial/\partial u}\frac{\partial}{\partial v} &= (\ln\mu)_v\frac{\partial}{\partial u} + \frac{4\mu\mu_u+\varphi\varphi_u}{4(K-\mu^2-1)-\varphi^2}\frac{\partial}{\partial v},\\
\nabla_{\partial/\partial v}\frac{\partial}{\partial v} &= -\frac{4\mu\mu_u+\varphi\varphi_u}{\mu^2(4(K-\mu^2-1)-\varphi^2)^2}\frac{\partial}{\partial u} + \frac{4\mu\mu_v+\varphi\varphi_v}{4(K-\mu^2-1)-\varphi^2}\frac{\partial}{\partial v}.
\end{aligned} \tag{5.109}$$

From (5.1), (5.3), (5.108), (5.109) and the formula of Gauss, we have

$$\begin{aligned}
L_{uu} &= \{iK + i\mu^2 + \frac{\mu_u}{\mu}\}L_u - \{4(K-\mu^2-1)-\varphi^2\}\mu\mu_v L_v,\\
L_{uv} &= \frac{\mu_v}{\mu}L_u + \left\{i\mu^2 + \frac{4\mu\mu_u+\varphi\varphi_u}{4(K-\mu^2-1)-\varphi^2}\right\}L_v\\
L_{vv} &= \left\{\frac{i}{4(K-\mu^2-1)-\varphi^2} - \frac{4\mu\mu_u+\varphi\varphi_u}{\mu^2(4(K-\mu^2-1)-\varphi^2)^2}\right\}L_u\\
&\quad + \left\{\frac{i\varphi}{\sqrt{4(K-\mu^2-1)-\varphi^2}} + \frac{4\mu\mu_v+\varphi\varphi_v}{4(K-\mu^2-1)-\varphi^2}\right\}L_v\\
&\quad - \frac{L}{4(K-\mu^2-1)-\varphi^2}.
\end{aligned}$$

A straightforward long computation shows that the compatibility conditions: $(L_{uu})_v = (L_{uv})_u$ and $(L_{uv})_v = (L_{vv})_u$ hold if and only if μ and φ satisfy

$$\mu_v = \frac{K\varphi_u + \varphi\mu\mu_u - \mu^2\varphi_u}{\mu(4(K-\mu^2-1)-\varphi^2)^{3/2}}, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G, \tag{5.110}$$

where $G = 1/\sqrt{4(K-\mu^2-1)-\varphi^2}$. From these we conclude that the Lagrangian surface is locally given by Case (28).

If $4(K-\mu^2-1) < \varphi^2$, (5.108) becomes

$$g = \mu^2\, \mathrm{d}u \otimes \mathrm{d}u + \frac{\mathrm{d}v \otimes \mathrm{d}v}{\varphi^2 - 4(K-\mu^2-1)}, \tag{5.111}$$

which yields

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial u}=\frac{\mu_u}{\mu}\frac{\partial}{\partial u}+\{4(K-\mu^2-1)-\varphi^2\}\mu\mu_v\frac{\partial}{\partial v},$$
$$\nabla_{\partial/\partial u}\frac{\partial}{\partial v}=\frac{\mu_v}{\mu}\frac{\partial}{\partial u}+\frac{4\mu\mu_u+\varphi\varphi_u}{4(K-\mu^2-1)-\varphi^2}\frac{\partial}{\partial v}, \tag{5.112}$$
$$\nabla_{\partial/\partial v}\frac{\partial}{\partial v}=\frac{4\mu\mu_u+\varphi\varphi_u}{\mu^2(4(K-\mu^2-1)-\varphi^2)^2}\frac{\partial}{\partial u}+\frac{4\mu\mu_v+\varphi\varphi_v}{4(K-\mu^2-1)-\varphi^2}\frac{\partial}{\partial v},$$

From (5.1), (5.3), and (5.112) we have

$$L_{uu}=\left\{i(K+\mu^2-1)+\frac{\mu_u}{\mu}\right\}L_u+(4(K-\mu^2-1)-\varphi^2)\mu\mu_v L_v,$$
$$L_{uv}=\frac{\mu_v}{\mu}L_u+\left\{i\mu^2+\frac{4\mu\mu_u+\varphi\varphi_u}{4(K-\mu^2-1)-\varphi^2}\right\}L_v,$$
$$L_{vv}=\left\{\frac{i}{4(K-\mu^2-1)-\varphi^2}+\frac{4\mu\mu_u+\varphi\varphi_u}{\mu^2(4(K-\mu^2-1)-\varphi^2)^2}\right\}L_u+\left\{\frac{i\varphi}{\sqrt{\varphi^2-4(K-\mu^2-1)}}+\frac{4\mu\mu_v+\varphi\varphi_v}{4(K-\mu^2-1)-\varphi^2}\right\}L_v+\frac{L}{4(K-\mu^2-1)-\varphi^2}.$$

A straightforward computation shows that the compatibility conditions: $(L_{uu})_v=(L_{uv})_u$ and $(L_{uv})_v=(L_{vv})_u$ hold if and only if μ and φ satisfy

$$\mu_v=\frac{\mu^2\varphi_u-K\varphi_u-\varphi\mu\mu_u}{\mu(\varphi^2-4(K-\mu^2-1))^{3/2}},\qquad \left(\frac{G_u}{\mu}\right)_u+\left(\frac{\mu_v}{G}\right)_v=-K\mu G, \tag{5.113}$$

where $G=1/\sqrt{\varphi^2-4(K-\mu^2-1)}$. From these we conclude that the Lagrangian surface is locally given by Case (29).

The converse can be verified by very long computations.

6. Some existence results

Proposition 1. *Let $\rho=\rho(u,v)$ and $\psi=\psi(u,v)$ be real-valued functions with $\rho_u,\rho_v,\psi_u,\psi_v\neq 0$ defined on a simply-connected open subset U of $\boldsymbol{R}^2$ satisfying*

$$\rho\rho_v=\psi\psi_u,\qquad \left(\frac{\rho_v}{\psi}\right)_u+\left(\frac{\rho_v}{\psi}\right)_v+\rho\psi=0. \tag{6.1}$$

Then $A_{\rho\psi} := (U, g_0)$ *with* $g_0 = \rho^2\, \mathrm{d}u \otimes \mathrm{d}u + \psi^2\, \mathrm{d}v \otimes \mathrm{d}v$ *is of constant curvature one. Moreover, up to rigid motions on* $CP^2(4)$, *there exists a unique Lagrangian isometric immersion* $\alpha_{\rho\psi} : A_{\rho\psi} \to CP^2(4)$ *whose second fundamental form satisfies*

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \qquad h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = J\frac{\partial}{\partial v}. \tag{6.2}$$

Proof. A direct computation shows that the Riemannian connection of $A_{\rho\psi}$ satisfies

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial u} = \frac{\rho_u}{\rho}\frac{\partial}{\partial u} - \frac{\psi_u}{\psi}\frac{\partial}{\partial v}, \qquad \nabla_{\partial/\partial u}\frac{\partial}{\partial v} = \frac{\rho_v}{\rho}\frac{\partial}{\partial u} + \frac{\psi_u}{\psi}\frac{\partial}{\partial v},$$
$$\nabla_{\partial/\partial v}\frac{\partial}{\partial v} = -\frac{\rho_v}{\rho}\frac{\partial}{\partial u} + \frac{\psi_v}{\psi}\frac{\partial}{\partial v}. \tag{6.3}$$

and $A_{\rho\psi}$ is of curvature one. If we define a symmetric bilinear form σ on $A_{\rho\psi}$ by

$$\sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u}, \qquad \sigma\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0, \qquad \sigma\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \frac{\partial}{\partial v}, \tag{6.4}$$

then (6.1), (6.3), (6.4) and the definitions of g_0, ρ, ψ imply that $\langle \sigma(X, Y), Z\rangle$ and $(\nabla\sigma)(X, Y, Z)$ are totally symmetric. Moreover, a direct computation shows that the curvature tensor R and σ satisfy condition (iii) of Theorem A. So, according to Theorems A and B, up to rigid motions there exists a unique Lagrangian immersion $\alpha_{\rho\psi} : A_{\rho\psi} \to CP^2(4)$ whose second fundamental form is given by (6.2). □

Proposition 2. *Let* $\mu = \mu(u, v)$ *and* $\Phi = \Phi(u, v)$ *be real-valued functions defined on a simply-connected open subset U of* $\boldsymbol{R}^2$ *satisfying*

$$\begin{aligned} &\frac{\partial \Phi}{\partial u} = \frac{\partial \mu^2}{\partial v} \neq 0, \\ &\left(\frac{\mu_v\sqrt{K-\mu^2-1}}{\Phi}\right)_v + \left(\frac{2(K-\mu^2-1)\mu_v + \Phi\mu_u}{(K-\mu^2-1)^{3/2}}\right)_u = \frac{-\mu\Phi K}{\sqrt{K-\mu^2-1}}, \end{aligned} \tag{6.5}$$

where K is a real number $\geq \mu^2$. *Then* $B^K_{\mu\Phi} := (U, g_1)$ *with*

$$g_2 = \mu^2\, \mathrm{d}u \otimes \mathrm{d}u + \frac{\Phi^2}{K-\mu^2-1}\, \mathrm{d}v \otimes \mathrm{d}v$$

is of constant curvature K. Moreover, up to rigid motions on $CP^2(4)$, *there exists a unique Lagrangian isometric immersion* $\beta^K_{\mu\Phi} : B^K_{\mu\Phi} \to CP^2(4)$ *whose second fundamental form satisfies*

$$\begin{aligned} &h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \left(K+\mu^2\right) J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \mu^2 J\frac{\partial}{\partial v}, \\ &h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \left(\frac{\Phi}{K-\mu^2-1}\right) J\frac{\partial}{\partial u} + 2\Phi J\frac{\partial}{\partial v}. \end{aligned} \tag{6.6}$$

Proof. A direct computation shows that the Riemannian connection of $P^K_{\mu\Phi}$ satisfies

$$\begin{aligned}\nabla_{\partial/\partial u}\frac{\partial}{\partial u} &= (\ln\mu)_u\frac{\partial}{\partial u} - \frac{(K-\mu^2-1)\mu\mu_v}{\Phi^2}\frac{\partial}{\partial v},\\ \nabla_{\partial/\partial u}\frac{\partial}{\partial v} &= (\ln\mu)_v\frac{\partial}{\partial u} + \left(\frac{\mu\mu_u}{K-\mu^2-1} + (\ln\Phi)_u\right)\frac{\partial}{\partial v},\\ \nabla_{\partial/\partial v}\frac{\partial}{\partial v} &= -\frac{\Phi^2\mu\mu_u + (K-\mu^2-1)\Phi\Phi_u}{\mu^2(K-\mu^2-1)^2}\frac{\partial}{\partial u}\\ &\quad + \left(\frac{\mu\mu_v}{K-\mu^2-1} + (\ln\Phi)_v\right)\frac{\partial}{\partial v}\end{aligned} \tag{6.7}$$

and the Gauss curvature of $P^K_{\mu\Phi}$ is the positive constant K.

If we define a symmetric bilinear form σ on $P^K_{\mu\Phi}$ by

$$\begin{aligned}&\sigma\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\right) = (K+\mu^2)\frac{\partial}{\partial u}, \qquad \sigma\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) = \mu^2\frac{\partial}{\partial v},\\ &\sigma\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right) = \left(\frac{\Phi}{K-\mu^2}\right)\frac{\partial}{\partial u} + 2\Phi\frac{\partial}{\partial v},\end{aligned} \tag{6.8}$$

then it follows from (6.5), (6.7), (6.8) and the definition of g_0 that $\langle\sigma(X,Y),Z\rangle$ and $(\nabla\sigma)(X,Y,Z)$ are totally symmetric. A direct computation shows that the curvature tensor R and σ satisfy condition (iii) of Theorem A. Thus, Theorems A and B imply that, up to rigid motions, there exists a unique Lagrangian immersion $\beta^K_{\mu\Phi}: B^K_{\mu\Phi} \to CP^2(4)$ whose second fundamental form is given by (6.6). □

Similarly, we have the following.

Proposition 3. *Let $\mu=\mu(u,v)$ and $\varphi=\varphi(u,v)$ be real-valued functions defined on a simply-connected open subset U of $\boldsymbol{R}^2$ satisfying*

$$\mu_v = \frac{K\varphi_u + \varphi\mu\mu_u - \mu^2\varphi_u}{\mu(4(K-\mu^2-1)-\varphi^2)^{3/2}} \neq 0, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G, \tag{6.9}$$

where $G = 1/\sqrt{4(K-\mu^2-1)-\varphi^2}$ for a real number $K > \mu^2+1+\varphi^2/4$. Then $C^K_{\mu\varphi} := (U, g_2)$ with $g_2 = \mu^2\,\mathrm{d}u\otimes\mathrm{d}u + G^2\,\mathrm{d}v\otimes\mathrm{d}v$ is of constant curvature K. Moreover, up to rigid motions, there exists a unique Lagrangian isometric immersion $\gamma^K_{\mu\varphi}: C^K_{\mu\varphi} \to CP^2(4)$ whose second fundamental form satisfies

$$\begin{aligned}&h\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\right) = (K+\mu^2+1)J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) = \mu^2 J\frac{\partial}{\partial v},\\ &h\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right) = \frac{1}{4(K-\mu^2-1)-\varphi^2}J\frac{\partial}{\partial u} + \frac{1}{\sqrt{4(K-\mu^2-1)-\varphi^2}}J\frac{\partial}{\partial v}.\end{aligned} \tag{6.10}$$

Proposition 4. *Let $\mu = \mu(u, v)$ and $\varphi = \varphi(u, v)$ be real-valued functions defined on a simply-connected open subset U of $\boldsymbol{R}^2$ satisfying*

$$\mu_v = \frac{\mu^2\varphi_u - K\varphi_u - \varphi\mu\mu_u}{\mu(\varphi^2 - 4(K-\mu^2-1))^{3/2}} \neq 0, \qquad \left(\frac{G_u}{\mu}\right)_u + \left(\frac{\mu_v}{G}\right)_v = -K\mu G, \tag{6.11}$$

where $G = 1/\sqrt{\varphi^2 - 4(K-\mu^2-1)}$ for a real number $K < \mu^2 + 1 + \varphi^2/4$. Then $D^K_{\mu\varphi} := (U, g_3)$ with metric $g_3 = \mu^2\,\mathrm{d}u \otimes \mathrm{d}u + G^2\,\mathrm{d}v \otimes \mathrm{d}v$ is of constant curvature K. Moreover, up to rigid motions, there exists a unique Lagrangian isometric immersion $\delta^K_{\mu\varphi} : D^K_{\mu\varphi} \to CP^2(4)$ whose second fundamental form satisfies

$$\begin{aligned} &h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = (K+\mu^2+1)J\frac{\partial}{\partial u}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \mu^2 J\frac{\partial}{\partial v},\\ &h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \frac{1}{\varphi^2 - 4(K-\mu^2-1)}J\frac{\partial}{\partial u} + \frac{1}{\sqrt{\varphi^2-4(K-\mu^2-1)}}J\frac{\partial}{\partial v}. \end{aligned} \tag{6.12}$$

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